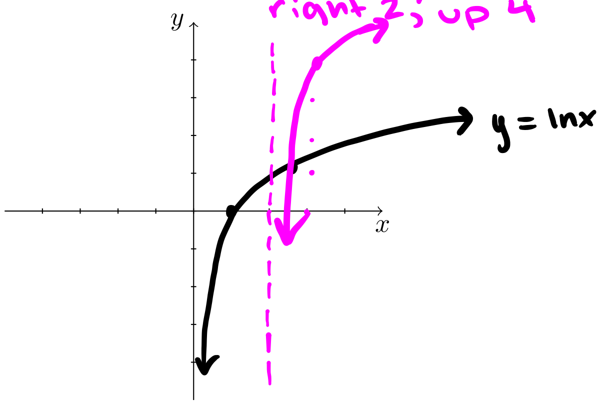
LECTURE: 1-3: NEW FUNCTIONS FROM OLD FUNCTIONS

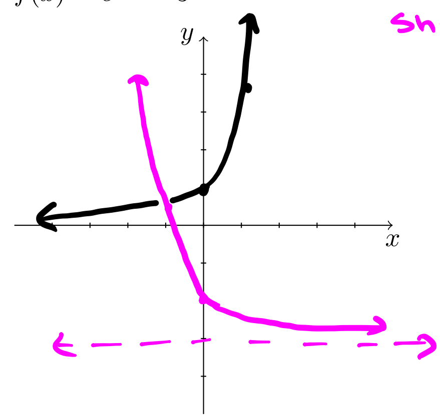
Example 1: Using transformations, sketch graphs of the following functions. Include a sketch of the parent function as well as the final graph of the given function.

(a) $f(x) = \ln(x-2) + 4$ shifts $y = \ln x$ right 2, up 4



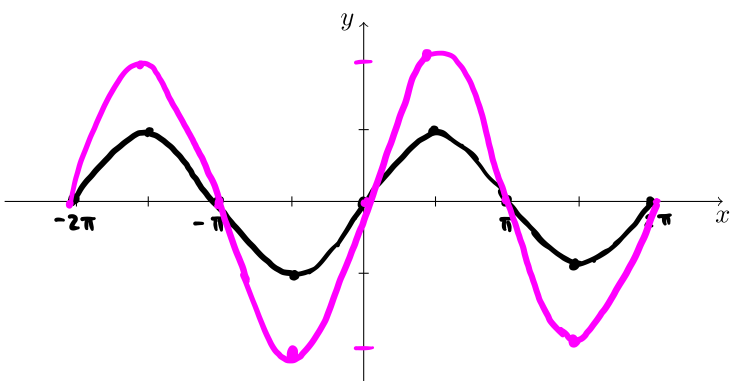
(b) $f(x) = e^{-x} - 3$

reflects $y = e^x$ over the y-axis, then shifts down 3.

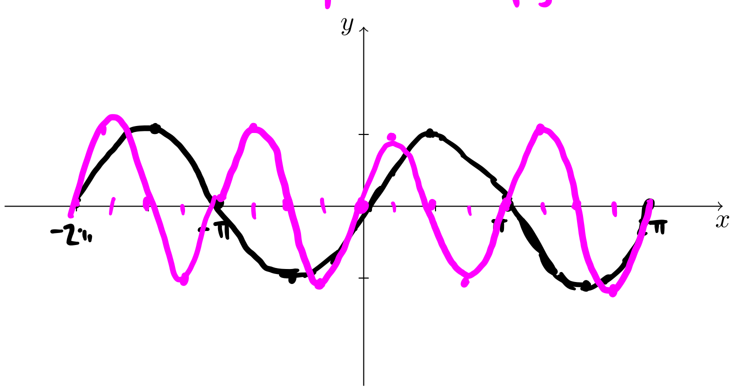


Example 2: Horizontal and vertical stretching and shrinking. Sketch graphs of the following functions on $[-2\pi, 2\pi]$. How do they relate to the parent function $f(x) = \sin x$?

(a) $g(x) = 2\sin x$ ← multiplies y-values of $y = \sin x$ by 2. This results in a vertical stretch!

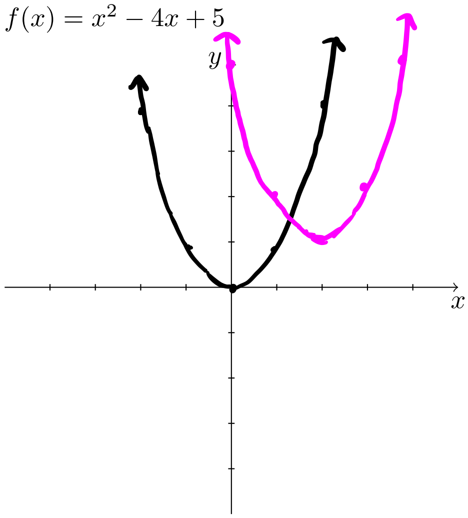


(b) $h(x) = \sin(2x)$ ← speeds up, oscillates twice as fast. In general, $\sin(ax)$ goes through a cycles in 2π.



Example 3: Review: completing the square and then using transformations. Use completing the square to write the following functions such that they can be graphed using transformations.

(a) $f(x) = x^2 - 4x + 5$



$f(x) = x^2 - 4x + 5$

$f(x) = x^2 - 4x + 4 + 5 - 4$

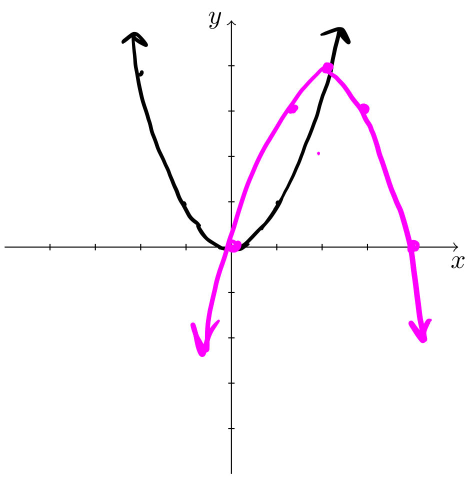
add ½ of this, squared

take it away so f is not changed.

$\boxed{f(x) = (x-2)^2 + 1}$

$y = x^2$ shifted right 2, up 1

(b) $f(x) = 4x - x^2$



$f(x) = -x^2 + 4x$

$f(x) = -(x^2 - 4x + 4) + 4$

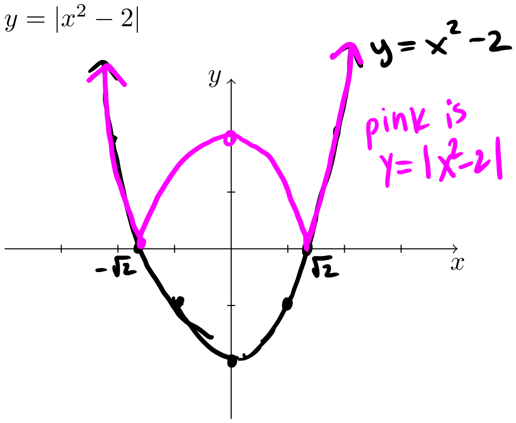
why +4?

$\boxed{f(x) = -(x-2)^2 + 4}$

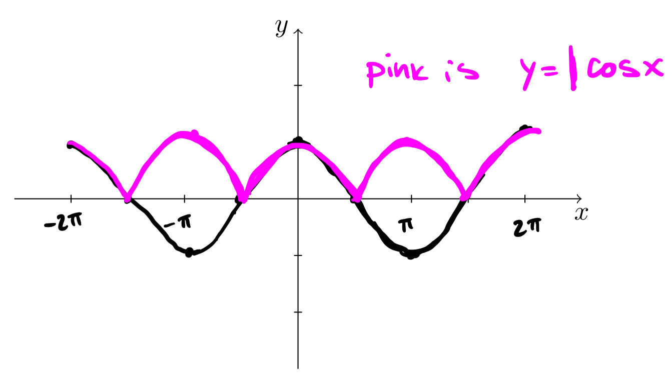
$y = x^2$ reflected over the x-axis, shifted right 2, up 4.

Example 4: How to deal with absolute values. Sketch the graphs of the following functions:

(a) $y = |x^2 - 2|$



(b) $y = |\cos x|$



To graph $y = |f(x)|$:

① graph $y = f(x)$

② all parts of $y = f(x)$ below the x-axis reflect over the x-axis. WHY?

Combinations of Functions

Example 5: If $f(x) = \sqrt{x}$ and $g(x) = \sqrt{4 - x^2}$, find the following functions and their domains.

(a) $(f + g)(x) = f(x) + g(x)$

$\boxed{= \sqrt{x} + \sqrt{4 - x^2}}$

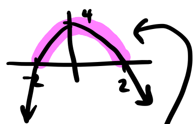
need: $\sqrt{x}$ to exist, thus $x \geq 0$.

and $\sqrt{4 - x^2}$ to exist, so

$4 - x^2 \geq 0$ ← Warning! Solving this is NOT like solving an equation.

$-2 \leq x \leq 2$

visual



only that part is positive

so, Domain is : $\boxed{0 \leq x \leq 2 \text{ or } [0,2]}$

(b) $(fg)(x) = f(x) \cdot g(x)$

$= \sqrt{x}\sqrt{4 - x^2}$

$= \boxed{\sqrt{4x - x^3}}$

$\boxed{D: [0, 2]}$

(same as (a))

(c) $(f/g)(x) = \dfrac{f(x)}{g(x)}$

$= \dfrac{\sqrt{x}}{\sqrt{4 - x^2}}$

$= \boxed{\sqrt{\dfrac{x}{4 - x^2}}}$

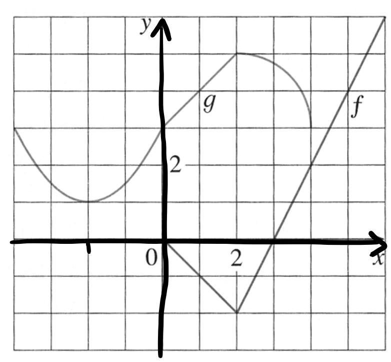
$\boxed{D: [0, 2)}$

Leave out things that make the denom. zero, in this case, 2.

Composition of Functions

> Given two functions f and g, the **composite function** $f \circ g$ is defined by
>
> $$(f \circ g)(x) = f(g(x)).$$
>
> Note: this is a **NEW** operation and is **NOT** the same as multiplying f and g.

Example 6: Use the graph below to find the following values or explain why it is undefined.



(a) $f(g(2)) = f(5) = \boxed{4}$

find $g(2)$ first, or the y-value of g when $x = 2$

(b) $(g \circ g)(-2) = g(g(-2)) = g(1) = \boxed{4}$

Example 7: If $f(x) = x^2$ and $g(x) = x - 3$, find the composite functions $f \circ g$ and $g \circ f$. Is it true that $f \circ g = g \circ f$?

$(f \circ g)(x) = f(g(x))$

$= f(x - 3)$

$= (x - 3)^2$

$= \boxed{x^2 - 6x + 9}$

$(g \circ f)(x) = g(f(x))$

$= g(x^2)$

$= \boxed{x^2 - 3}$

note that $(f \circ g)(x) \neq (g \circ f)(x)$. Thus, $f \circ g \neq g \circ f$, in general. We say that function composition is a non commutative operation.

Example 8: If $f(x) = \cos x$ and $g(x) = 1 - \sqrt{x}$ find the following and their domains.

(a) $(f \circ g)(x) = f(g(x))$

$= f(1-\sqrt{x})$

$= \boxed{\cos(1-\sqrt{x})}$

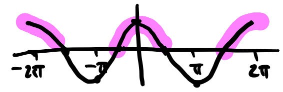
note $x \geq 0$ because of $\sqrt{x}$, you can cosine anything, so Domain: $\boxed{x \geq 0 \text{ or } [0, \infty)}$

(b) $(g \circ f)(x) = g(f(x))$

$= g(\cos x)$

$= \boxed{1 - \sqrt{\cos x}}$

we need $\cos x \geq 0$, this is hard, let's look at a picture



$\boxed{D: \ldots [-\tfrac{5\pi}{2}, -\tfrac{3\pi}{2}] \cup [-\pi/2, \pi/2] \cup [3\pi/2, 5\pi/2] \cup \ldots}$

Example 9: Find $f \circ g \circ h$ if $f(x) = 2/(x+1)$, $g(x) = \cos x$ and $h(x) = \sqrt{x+3}$.

$f(g(h(x))) = f(g(\sqrt{x+3}))$

$= f(\cos(\sqrt{x+3}))$

$= \boxed{\dfrac{2}{\cos(\sqrt{x+3})+1}}$

Example 10: What were those functions? Given the following compositions find, f, g and h such that $F = f \circ g \circ h$.

3rd 2nd 1st

(a) $F(x) = \cos^2(x+9) = (\cos(x+9))^2$

$\boxed{\begin{aligned} f(x) &= x^2 \\ g(x) &= \cos x \\ h(x) &= x+9 \end{aligned}}$

(b) $F(x) = \tan^4(x^2+1)$

$= (\tan(x^2+1))^4$

$\boxed{\begin{aligned} f(x) &= x^4 \\ g(x) &= \tan x \\ h(x) &= x^2+1 \end{aligned}}$ OR $\boxed{\begin{aligned} f(x) &= x^4 \\ g(x) &= \tan(x+1) \\ h(x) &= x^2 \end{aligned}}$

OR others!

Example 11: Suppose g is an even function and let $h = f \circ g$. Is h also an even function?

Try inputting $-x$: $h(-x) = (f \circ g)(-x)$

$= f(g(-x))$ because g is even, $g(-x) = g(x)$

$= f(g(x))$

$= (f \circ g)(x) = h(x)$; so $\boxed{\text{Yes}}$, $\boxed{h \text{ is even}}$

Example 12: Let f and g be linear functions with equations $f(x) = m_1x + b_1$ and $g(x) = m_2x + b_2$. Is $f \circ g$ also a linear function? If so, what is the slope of its graph? What is its y-intercept?

$(f \circ g)(x) = f(g(x))$

$= f(m_2x + b_2)$

$= m_1(m_2x + b_2) + b_1$

$= m_1m_2x + m_1b_2 + b_1$

$= mx + b$

where $\boxed{\begin{aligned} m &= m_1m_2 \text{ (slope)} \\ b &= m_1b_2 + b_1 \text{ (y-int)} \end{aligned}}$ and

$\boxed{\text{so Yes! } f \circ g \text{ is linear}}$