LECTURE: 1-4: EXPONENTIAL FUNCTIONS

Example 1: Suppose you are offered a job that lasts one month. Which of the following methods of payment do you prefer?

(a) One million dollars at the end of the month.

(b) One cent the first day, two cents the second, four cents the third, etc.

Laws of Exponents If a and b are positive numbers and x and y are real numbers, then

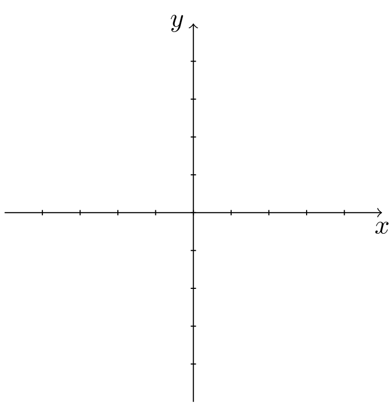
(a) $b^x b^y =$________ (b) $\frac{b^x}{b^y} =$ ________ (c) $(b^x)^y =$ ________ (d) $(ab)^x =$ ________

Example 2: Use the laws of exponents to simplify the following expressions.

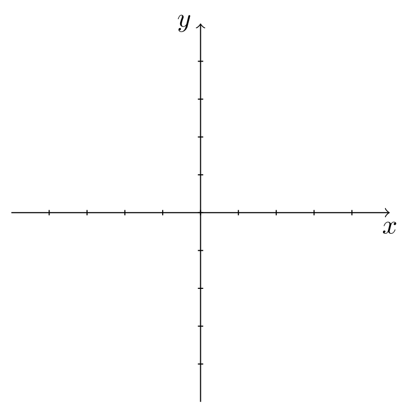
(a) $e^2 e^x$ (b) $(e^{5x})^2$ (c) $\frac{5^2}{5^x}$

Example 3: Graph the following exponential functions.

(a) $f(x) = 5 - e^x$



(b) $f(x) = (1/2)^x$



Example 4: Find the exponential function $f(x) = a \cdot b^x$ who passes through the points $(1, 6)$ and $(3, 24)$.

Example 5: The half-life of strontium-90 is 25 years, meaning half of any given quantity of strontium-90 will disintegrate in 25 years.

(a) If a sample of strontium-90 has a mass of 100 mg, find an expression for the mass $m(t)$ that remains after t years.

(b) Find the mass remaning after 40 and 80 years.

(c) Estimate the time required for the mass to be reduced to 5 mg.

Inverse Functions

Generally speaking, inverse functions are functions that "undo" one another. For example, if I square a number, to undo this operation I take a square root. Thus, $f(x) = x^2$ and $g(x) = \sqrt{x}$ are inverse functions. There are some technicalities to this relationship, but the basic idea that inverses "undo" each other is a good place to start.

Defintion: A function f is called **one-to one** if it never takes on the same values twice. That is,

$$f(x_1) \neq f(x_2) \quad \text{whenever} \quad x_1 \neq x_2$$

Horizontal Line Test A function f is one-to-one if and only if no horizontal line intersects its graph more than once.

Example 6: Are the following functions one-to-one?

(a) $f(x) = x^3$

(b) $f(x) = x^2$

(c) $f(x) = e^x$

Definition: Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A. It is defined by

$$f^{-1}(y) = x \text{ if and only if } f(x) = y$$

for any y in B.

One consequence of the definition above are the following cancellation equations.

- $f(f^{-1}(x)) = x$ for all x in B
- $f^{-1}(f(x)) = x$ for all x in A

Example 7: If $f(1) = 5$, $f(3) = 7$, and $f(8) = -10$ find the following.

(a) $f^{-1}(7)$

(b) $f^{-1}(5)$

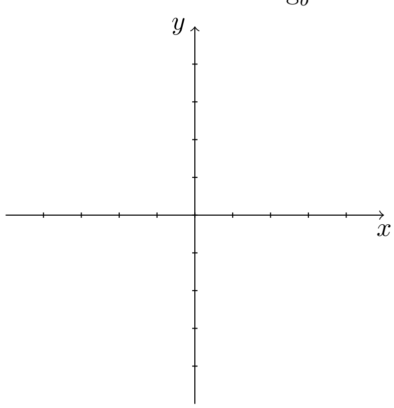
Example 8: Find the inverse of the following functions. Give the domain and range of the inverse.

(a) $f(x) = (x+2)^3 - 5$

(b) $f(x) = \dfrac{2x+3}{x-5}$

Logarithmic Functions

If $b \neq 1$, the exponential function $f(x) = b^x$ is either increasing or decreasing is therefore one-to-one by the Horizontal Line Test. Thus, this function has an inverse function which we call the **logarithmic function with base** b and is denoted $\log_b x$. For $b = e$ sketch a graph of $f(x) = e^x$ and $f^{-1}(x) = \log_e x = \ln x$.



As the functions $f(x) = b^x$ and $g(x) = \log_b x$ are inverses, we have the cancellation equations. ($\log_e x = \ln x$.)

a) $f(g(x)) =$ ______________________________ for every _______________

b) $g(f(x)) =$ ______________________________ for every _______________

Example 10: Find the exact values of the following expressions.

a) $\log_5 125$

b) $\ln e^5$

c) $\ln \frac{1}{e^2}$

Laws of Logarithms If x and y are positive numbers, then

1. $\log_b(xy) = \log_b x + \log_b y$
2. $\log_b(x/y) = \log_b x - \log_b y$
3. $\log_b(x^r) = r \log_b x$

Example 11: Use properties of logarithms to express the following quantities as one logarithm (a) and expand the logarithm in (b).

(a) $\log b + 2 \log c - 3 \log d$

(b) $\ln\left(\frac{\sqrt{x^2+5}(x-3)^5}{(x+5)^2}\right)$

Example 12: Solve the following equations for x.

(a) $\ln(x+5) - 1 = 7$

(b) $e^{2x-5} + 4 = 10$

Example 13: Find the domain of the following functions.

(a) $f(x) = \dfrac{1 - e^{x^2}}{1 - e^{1-x^2}}$

(b) $g(x) = \sqrt{e^x - 2}$

Common Mistakes and Misconceptions

Example 14: Are the following statements true or false? If either case, explain why. If possible, change the false statements so that they are a true statement.

(a) $(a+b)^2 = a^2 + b^2$

(b) $\sqrt{x^2+4} = x + 2$

(c) $\dfrac{a+b}{c+d} = \dfrac{a}{c} + \dfrac{b}{d}$

(d) $\dfrac{a+b}{c} = \dfrac{a}{c} + \dfrac{b}{c}$

(e) $\ln(x+y) = \ln x + \ln y$

(f) $\dfrac{\ln x}{\ln y} = \ln\left(\dfrac{x}{y}\right)$

(g) $\ln(x-y) = \ln\left(\dfrac{x}{y}\right)$

(h) $f^{-1}(x) = \frac{1}{f(x)}$

(i) $f^2(x) = (f(x))^2$