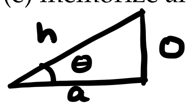
Lecture: 1-5: Trigonometry Review and Inverse Trigonometric Functions

Basic Trigonometry

In this class you are **expected** to have some basic understanding and proficiency in trigonometry. Specifically, you should know (a) the triangle definitions of all six trigonometric functions, (b) the definitions of the four non-sine and cosine trigonometric functions in terms of sine and cosine, (c) be able to graph all six trigonometric functions, (d) be familiar with the unit circle definition and be able to evaluate all trigonometric functions at common angles without the use of a calculator, (e) memorize and be able to use six basic, trigonometric identities.

The Triangle Defintion



soh – cah – toa

Example 1: Sketch a right triangle with side a adjacent to an angle θ, o opposite of the angle θ and hypotenuse h. Define each of the six trigonometric functions in terms of that triangle.

a) $\sin\theta = \frac{o}{h} = \frac{\text{opp}}{\text{hyp}}$

b) $\cos\theta = \frac{a}{h} = \frac{\text{adj}}{\text{hyp}}$

c) $\tan\theta = \frac{o}{a} = \frac{\text{opp}}{\text{adj}}$

d) $\sec\theta = \frac{h}{a}$ (recip of $\cos\theta$) $= \frac{\text{hyp}}{\text{adj}}$

e) $\csc\theta = \frac{h}{o}$ (recip of $\sin\theta$) $= \frac{\text{hyp}}{\text{opp}}$

f) $\cot\theta = \frac{a}{o}$ (recip of $\tan\theta$) $= \frac{\text{adj}}{\text{opp}}$

Functions in Terms of Sine and Cosine

Example 2: Define the following four functions in terms of sine and cosine. How does this relate to your answers to Example 1?

(a) $\tan\theta = \frac{\sin\theta}{\cos\theta}$ $\quad \frac{(o/h)}{(a/h)} = (o/a)$

(b) $\sec\theta = \frac{1}{\cos\theta}$

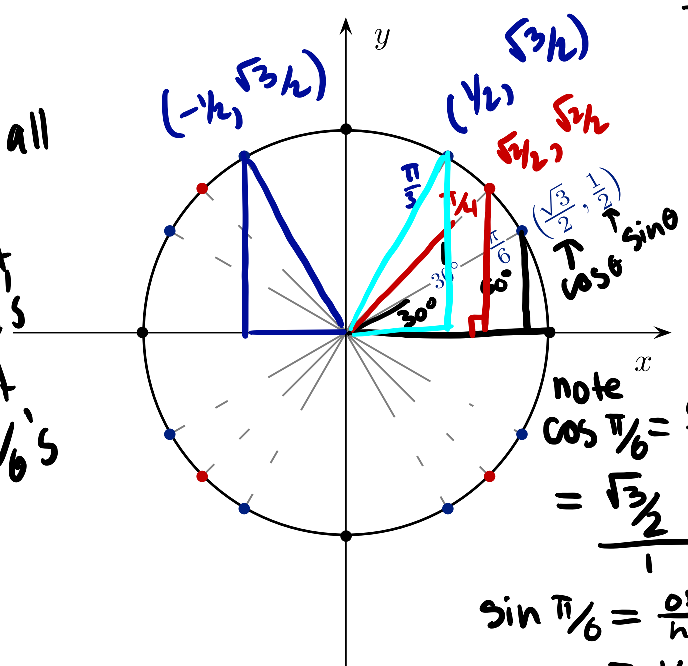
(c) $\csc\theta = \frac{1}{\sin\theta}$

(d) $\cot\theta = \frac{\cos\theta}{\sin\theta}$

The Unit Circle Approach

Example 3: Using the 45-45-90 triangle and 30-60-90 triangle find the coordinates on the unit circle in Quadrant 1. What pattern do you see? What coordinate on the unit circle gives sine? What coordinate gives cosine? Does this agree with the triangle definition of sine and cosine? Now, reflect the points in quadrant 1 into quadrant 2.

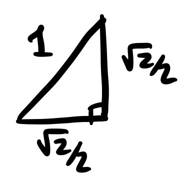
to get all angles:
① count by $\pi/4$'s
② count by $\pi/6$'s



note $\cos\pi/6 = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{3}/2}{1}$

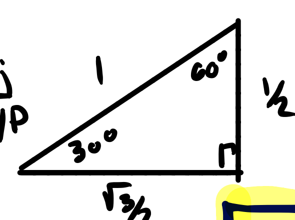
$\sin\pi/6 = \frac{\text{opp}}{\text{hyp}} = 1/2$

The 45-45-90 triangle



(can verify using pythagorean theorem)

The 30-60-90 triangle

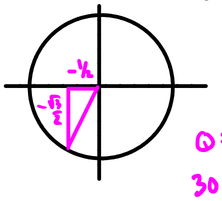


(ditto on pythagorean thm)

x-coord ⇒ cosine
y-coord ⇒ sine

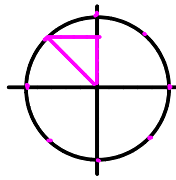
Example 4: Using the unit circle idea, find sine and cosine of the following angles. Don't look back at page 1. Rather, sketch a small unit circle and ask yourself these questions (1) what quadrant?, (2) what triangle or axis?, (3) are you looking for x (horizontal) or y (vertical)?

(a) $\sin(-\frac{2\pi}{3}) = \boxed{-\frac{\sqrt{3}}{2}}$



Q3, y-coord.
30-60-90

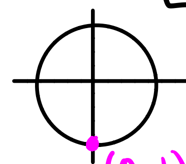
(b) $\cos(\frac{11\pi}{4}) = \boxed{-\frac{\sqrt{2}}{2}}$



Hint: If you see some $\pi/4$ multiple it's always $\pm \sqrt{2}/2$!

Q2, x-coord, 45-45-90

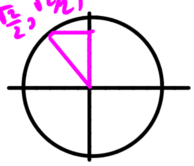
(c) $\cos(\frac{3\pi}{2}) = \boxed{0}$



(0,1)

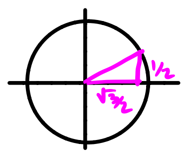
Example 5: Find the following values.

(a) $\tan(\frac{3\pi}{4}) = \frac{\sin 3\pi/4}{\cos 3\pi/4} = \frac{(\sqrt{2}/2)}{(-\sqrt{2}/2)} = \boxed{-1}$



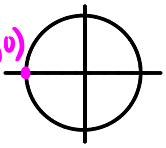
$(-\frac{\sqrt{2}}{2}, \sqrt{2}/2)$

(b) $\cot(\frac{\pi}{6}) = \frac{\cos \pi/6}{\sin \pi/6} = \frac{\sqrt{3}/2}{1/2} = \boxed{\sqrt{3}}$



(c) $\sec(\pi) = \frac{1}{\cos\pi} = \frac{1}{-1} = \boxed{-1}$

x-coord

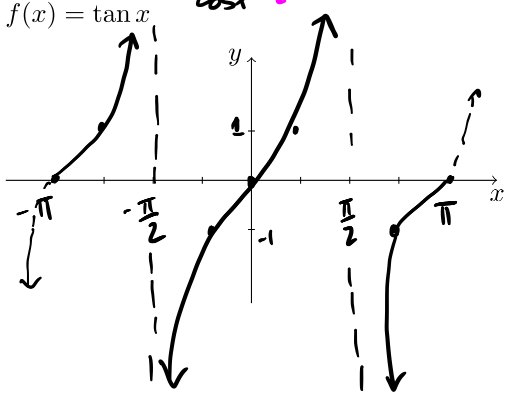


(-1,0)

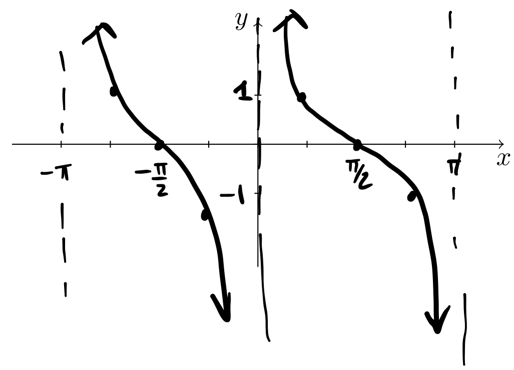
x-coord

Example 6: Graph the following functions on $[-\pi, \pi]$. Clearly label any asymptotes.

a) $f(x) = \tan x = \frac{\sin x}{\cos x}$ — $\cos x$ is 0 at $\pm\pi/2$

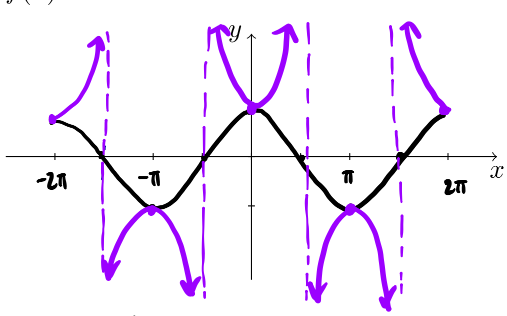


b) $f(x) = \cot x = \frac{\cos x}{\sin x}$ — $\sin x$ is zero at $0, \pm\pi$



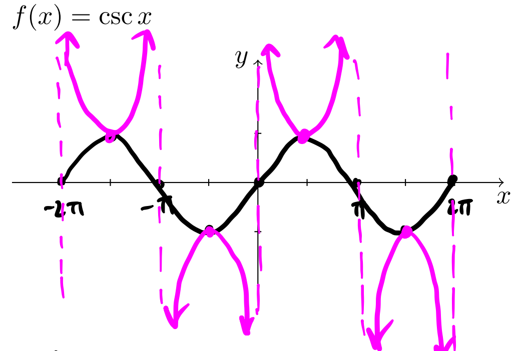
Example 7: Graph the following functions on $[-2\pi, 2\pi]$. Clearly label any asymptotes.

a) $f(x) = \sec x$



① $\sec x = 1/\cos x$, graph $y = \cos x$
② sketch in asymptotes
③ Sketch in curve.

b) $f(x) = \csc x$



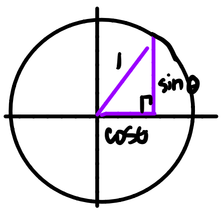
① $\csc x = 1/\sin x$

Common Identities

When you first learned trigonometry you probably had to prove a bunch of equivalencies using various identities. Perhaps you discussed why these identies were true. Some of you may have been asked to memorize all of these and some of you may have been allowed a note card/ cheat sheet. In this class you need to have the following **SIX** identities and no more.

The Pythagorean Identities: Looking at the unit circle, derive an identity involving sine and cosine using the pythagorean theorem. This is (a). Next, divide the answer to (a) by $\cos^2\theta$ to obtain a new identity for (b). Finally, divide (a) by $\sin^2\theta$ to obtain the third Pythagorean identity for (c).

(a) Identity #1



so, $a^2+b^2=c^2$

$\boxed{\cos^2\theta+\sin^2\theta=1}$

(b) Identity #2 (÷(a) by $\cos^2\theta$)

$\frac{\cos^2\theta}{\cos^2\theta}+\frac{\sin^2\theta}{\cos^2\theta}=\frac{1}{\cos^2\theta}$

$\boxed{1+\tan^2\theta=\sec^2\theta}$

(c) Identity #3 (÷ (a) by $\sin^2\theta$)

$\frac{\cos^2\theta}{\sin^2\theta}+\frac{\sin^2\theta}{\sin^2\theta}=\frac{1}{\sin^2\theta}$

$\boxed{\cot^2\theta+1=\csc^2\theta}$

The Half/ Double Angle Identities: You have probably seen the identities $\cos(2\theta)=1-2\sin^2\theta$ and $\cos(2\theta)=1+2\cos^2\theta$. It is more likely that we will use these slightly differently in this class. Solve these identities for $\sin^2\theta$ and $\cos^2\theta$ respectively. Finally, given an identity for $\sin(2\theta)$.

(a) $\cos(2\theta)=1-2\sin^2\theta$

$\cos 2\theta-1=-2\sin^2\theta$

$-\frac{1}{2}(\cos 2\theta-1)=\sin^2\theta$

$\boxed{\sin^2\theta=\frac{1}{2}(1-\cos 2\theta)}$

(b) $\cos(2\theta)=$ $2\cos^2\theta-1$

$\cos 2\theta+1=2\cos^2\theta$

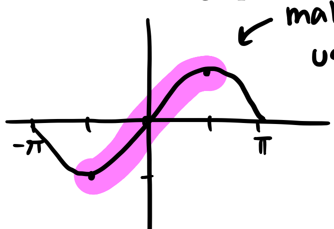
$\frac{1}{2}(1+\cos 2\theta)=\cos^2\theta$

$\boxed{\cos^2\theta=\frac{1}{2}(1+\cos 2\theta)}$

(c) $\sin(2\theta)=\boxed{2\sin\theta\cos\theta}$

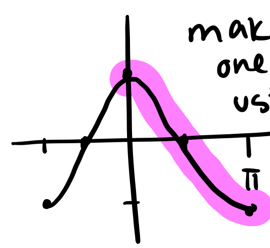
Inverse Trigonometric Functions

Definition and graphs of $\sin^{-1}x$ and $\cos^{-1}x$



make sinx one-to-one using $[-\pi/2, \pi/2]$ D

$\sin^{-1}x$ asks what angle in gives x?



make cosx one to one using $[0,\pi]$

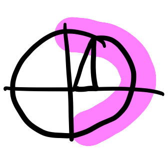
$\cos^{-1}x$ says what angle in gives x when cosined?

Example 9: Evaluate the following expressions.

(a) $\sin^{-1}(\sqrt{3}/2)$ $=\pi/3$

$\sin x=\sqrt{3}/2$

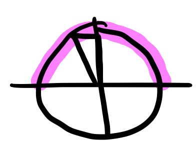
in $[-\pi/2, \pi/2]$?



(b) $\cos^{-1}(-1/2)=\boxed{2\pi/3}$

$\cos x=-1/2$

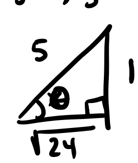
in $[0,\pi]$?



(c) $\tan(\arcsin 1/5)$ $=\tan(\theta)$

$\arcsin 1/5=\theta$

so, $1/5=\sin\theta$



$=\frac{1}{\sqrt{24}}$

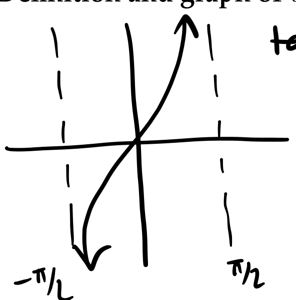
$=\sqrt{24}/24$

$=2\sqrt{6}/24$

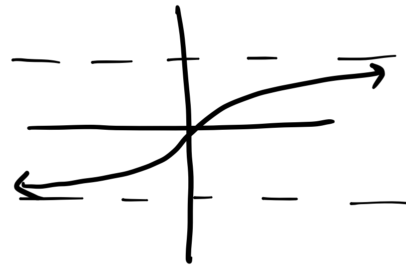
$=\boxed{\sqrt{6}/12}$

use the triangle to get tanθ

Definition and graph of $\tan^{-1} x$



tanx is one to one on $-\pi/2, \pi/2$ $\Rightarrow$



$y = \tan^{-1} x$

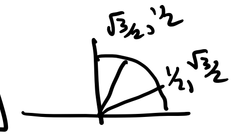
D: $\mathbb{R}$

R: $(-\pi/2, \pi/2)$

Example 10: Simplify the following expressions.

a) $\boxed{\tan^{-1}(-1) = -\pi/4}$ b) $\tan^{-1}(\sqrt{3}) = \boxed{\pi/6}$ c) $\tan^{-1} 0 = \boxed{0}$ (see on the graph that $\tan^{-1}x$ goes through $(0,0)$)

$\tan x = -1$ in $(-\pi/2, \pi/2)$? $\tan x = \sqrt{3}$ in $(-\pi/2, \pi/2)$?

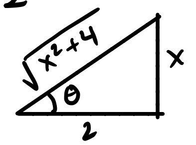


Example 11: Simplify the following expressions. $\leftarrow$ tricky triangles

a) $\sin(\tan^{-1} \frac{x}{2}) = \sin\theta$ (opp/hyp) $= \boxed{\dfrac{x}{\sqrt{x^2+4}}}$

$\tan^{-1}\frac{x}{2} = \theta$,

$\frac{x}{2} = \tan\theta$



b) $\sin(2 \arccos x)$ (use $\sin 2\theta = 2\sin\theta\cos\theta$ identity)

$= 2\sin(\arccos x)\cos(\arccos x)$

$= 2\sin(\cos^{-1}x)\cdot x$

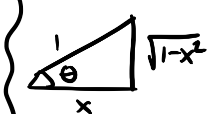
tricky triangle $\rightarrow$

$= 2\sin(\theta)\cdot x$

$= 2\cdot\sqrt{1-x^2}\cdot x = \boxed{2x\sqrt{1-x^2}}$

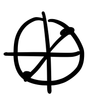
$\cos^{-1}x = \theta$

$\frac{x}{1} = \cos\theta$



Example 12: How is $\tan^{-1}(1)$ different from solving the equation $\tan x = 1$? Why does the former have only one value while the latter has an infinite number of values?

- $\tan^{-1}(1) = \pi/4$; this has ONE output because $\tan^{-1}x$ is a function
- solving $\tan x = 1$, we want all values where $\tan x = 1$



$x = \pi/4, 3\pi/4, 5\pi/4, \ldots$ or $\boxed{x = \pi/4 + n\pi}$ where n is an integer

Example 13: Are the following statements true or false?

(a) $\sin^{-1} x = \frac{1}{\sin x}$ (b) $\tan^{-1}(-1) = \frac{3\pi}{4}, \frac{7\pi}{4}$ (c) $\sin^2 x = 1 + \cos^2 x$

False, that (-1) does not mean a power of (-1); it means "inverse" function of sine.

False, $\tan^{-1}(-1)$ sol. to is $\tan x = -1$ in $(-\pi/2, \pi/2)$ or $\boxed{-\pi/4}$, also $\tan^{-1}x$ is a function + can't have two outputs.

False, $\sin^2 x + \cos^2 x = 1$ so, $\sin^2 x = 1 - \cos^2 x$ but not $1 + \cos^2 x$.