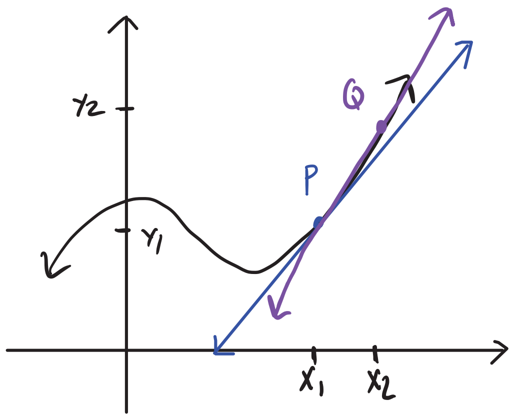
Lecture Notes 2-1: The Tangent and Velocity Problems

The importance of a good question.

Question 1: Given the graph of a function $y = f(x)$ and a point P on this graph, how do you *define* and *find* the equation of the tangent line to the graph at P?



The problem: to find an equation of a line you need TWO points, you only have one.

The solution: ESTIMATE! Pick a nearby point, call it Q. Find the equation of this line. To make a better estimate pick Q closer to P.

Note if $Q \to P$, $x_2 \to x_1$

Question 2: Given the position of an object (say a cell phone) at any time, how do you *define* and *find* the velocity of the object at a particular instant (say the moment your child launches it off a cliff)?

suppose I know the position of an object (in feet), how do I find its velocity (in ft/sec) at any instant in time?

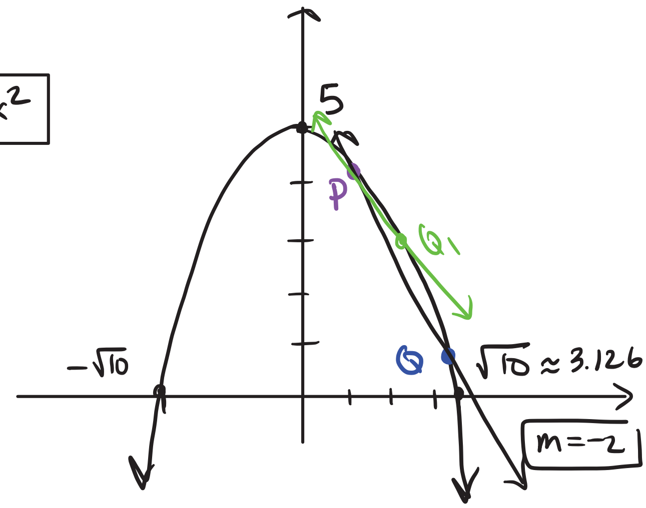
I can find $\frac{\text{change in position}}{\text{change in time}}$ or $\frac{\Delta \text{ position}}{\Delta t}$, and

as $\Delta t \to 0$ this expression approaches the "true" instantaneous velocity.

Some Facts:

- These questions are old. (200BC or older depending on your interpretation)
- These questions are hard, taking more than a thousand years and untold numbers of mathematicians to answer.
- Before finding solid mathematical ground, some of its ideas were even more controversial than Donald Trump's tweets are today!
- Attempts to answer these two questions is *part* of what led to the development/discovery of Calculus. (ie. Newton + Leibniz)
- The ideas you learn in calculus explain planetary motion or where a projectile will land or predict how fast an infection will spread.
- **Most importantly and perhaps obviously,** *the questions that motivated the development of calculus go a long way to explaining the definitions and applications we see later*

Example 1: Let $f(x) = (10 - x^2)/2 = \boxed{5 - \frac{1}{2}x^2}$



(a) Sketch a LARGE graph of $f(x)$ in the space to the right. Include any x- or y-intercepts.

$0 = (10 - x^2)/2$

$0 = 10 - x^2$

$x^2 = 10$

$x = \pm\sqrt{10}$ (x-int)

(b) Let P be the point on the curve where $x = 1$ and let Q be the point on the curve where $x = 3$. Find the y-coordinate for P and Q and plot them on your graph above.

$x = 1, \quad f(1) = (10-1)/2 = 9/2 = 4.5 \Rightarrow \boxed{P = (1, 4.5)}$

$x = 3, \quad f(3) = (10-9)/2 = 1/2 \Rightarrow \boxed{Q = (3, 0.5)}$

(c) DEFINITION: A *secant line* on a graph is simply the line determined by two points on the graph. Find the EQUATION of the secant line determined by the points P and Q and graph it above.

$$\text{slope} = m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0.5 - 4.5}{3 - 1} = \frac{-4}{2} = \boxed{-2}$$

$y - y_1 = m(x - x_1)$

$y - 4.5 = -2(x - 1)$

$y - 4.5 = -2x + 2$

$\boxed{y = -2x + 6.5}$

(d) Label the line you just plotted above with its *slope.*

these are getting closer to 1.

(e) For the FIVE points Q_1, Q_2, Q_3, Q_4, Q_5 with x-coordinates 2, 1.5, 1.25, 1.125, 1.0625, find the y-coordinate, plot the point, plot the secant line determined by P and Q_i, and label the line with its slope.

$\boxed{f(x) = (10 - x^2)/2}$ that's how to get these

x	2	1.5	1.25	1.125	1.0625
$y = f(x)$	3	3.875	4.219	4.367	4.4355
$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_2 - 4.5}{x - 1}$	−1.5	−1.25	−1.125	−1.0625	−1.0325

(1, 4.5) is our fixed point

Note as $Q \to P$ we are getting closer to our tangent line.

(f) Sketch what YOU think the tangent to $f(x)$ at the point P should look like...???

(g) What do you observe about the relationship between the secant lines you **calculated** and the tangent line you **guessed at**?

The slopes of the secant line are approaching the slope of the tangent line.

(h) What is the significance of the **words in bold** in the previous question?

we really want the slope of the tangent line, but we're forced to calculate secant lines because we need two points.

(i) What PART of the tangent line is indicated by the sequence of secant lines? $\boxed{\text{slope}}$

(j) Write the *equation* of the tangent line to $f(x)$ at P. Does this answer seem reasonable? Why or why not?

Point P is (1, 4.5)

guess $m = -1$

$y - y_1 = m(x - x_1)$

$y - 4.5 = -1(x - 1)$

$y - 4.5 = -x + 1$

$\boxed{y = -x + 5.5}$

(k) In *plain old ENGLISH SENTENCES* how would you explain (step-by-step) how to find the *equation* of the tangent line?

1. choose points on the curve that are approaching P
2. Find slopes of the secant lines determined by P + points from #1.
3. make a guess (an educated one) at m based on what happens as your points approach P.
4. write an equation using P + m from #3

(l) In the previous exercise, we chose points (Q_i's) on the *right* of the point P, what would happen if we had chosen points on the *left*?

The slopes still would have approched -1.