Lecture Notes 2-3: Calculating Limits Using the Limit Laws

Review:

1. What does it mean to write $\lim_{x\to a} f(x) = L$?

as x gets close to a, f(x) gets close to L.

2. Given $f(x)$ and a how did we find the limit L or show that it doesn't exist?

- plug in numbers getting closer to a and made a guess
- look @ a graph of $y=f(x)$

Goals:

- Learn a whole bunch of *general principles* about calculating limits.
- Correct and careful application of these principles will allow us to (a) avoid the tedious calculation from Section 2-2 and (b) avoid the mistakes and pitfalls of relying on numerical approximations.

Limit Laws (Table 1)

Don't forget that these rules are true only if these hypotheses are met.

In the rules below c is a constant and $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist.

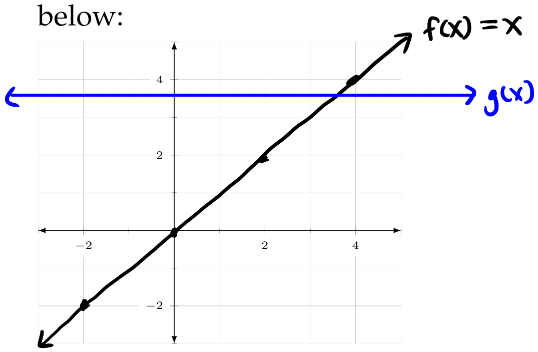
formal notation	in English sentences
1. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$	1. limits pass through sums limit of a sum is sum of limits
2. $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$	2. limits pass through differences
3. $\lim_{x\to a}[cf(x)] = c \lim_{x\to a} f(x)$	3. multiplied constants can be pulled out of limits
4. $\lim_{x\to a}[f(x)\cdot g(x)] = \left[\lim_{x\to a} f(x)\right] * \left[\lim_{x\to a} g(x)\right]$	4. limits pass through products
5. $\lim_{x\to a}[f(x)/g(x)] = \left(\lim_{x\to a} f(x)\right) / \left(\lim_{x\to a} g(x)\right)$	5. limits pass through quotients. (provided denom does not limit to zero.)

* provided $\lim_{x\to a} g(x) \neq 0$.

why is this needed??

EXAMPLE 1:

1. Graph $f(x) = x$ and $g(x) = 3.8$ on the axes below:



2. Use the graphs to evaluate the limits below:

$\lim_{x\to 2} f(x) =$ 2

$\lim_{x\to 2} g(x) =$ 3.8

3. Do the limits $\lim_{x\to 2} f(x)$ and $\lim_{x\to 2} g(x)$ exist? Why?

Yes, we found them graphically. The limits from the left & right at x=2 are equal.

4. Use the limit laws and part 2. above to evaluate $\lim_{x\to 2}[8x - 3.8]$. Justify your steps.

$$\lim_{x\to 2}[8x - 3.8] = \lim_{x\to 2} 8x - \lim_{x\to 2} 3.8 \quad \text{(Rule 2)}$$
$$= 8\lim_{x\to 2} x - \lim_{x\to 2} 3.8 \quad \text{(Rule 3)}$$
$$= 8(2) - 3.8$$
$$= 16 - 3.8 = \boxed{12.2}$$

5. Use the limit laws and part 2. above to evaluate $\lim_{x\to 2} x^5$. Justify your steps.

$$\lim_{x\to 2} x^5 = \lim_{x\to 2}(x \cdot x \cdot x \cdot x \cdot x)$$
$$= \left(\lim_{x\to 2} x\right)\left(\lim_{x\to 2} x\right)\left(\lim_{x\to 2} x\right)\left(\lim_{x\to 2} x\right)\left(\lim_{x\to 2} x\right) \quad \text{(Rule 4)}$$
$$= 2^5$$
$$= \boxed{32}$$

Limit Laws (Table 2)

In the rules below c is a constant, n is a positive integer, and $\lim_{x\to a} f(x)$ exists.

1. $\lim_{x\to a}(f(x))^n =$ $\left[\lim_{x\to a} f(x)\right]^n$	2. $\lim_{x\to a} c =$ c
3. $\lim_{x\to a} x =$ a	4. $\lim_{x\to a} x^n =$ a^n
5. $\lim_{x\to a} \sqrt[n]{x} =$ $\sqrt[n]{\lim_{x\to a} x} = \sqrt[n]{a}$	6. $\lim_{x\to a} \sqrt[n]{f(x)} =$ $\sqrt[n]{\lim_{x\to a} f(x)}$

EXAMPLE 2: Evaluate $\lim_{x\to -3} \frac{\sqrt{x^2-5}}{4-2x}$ and justify your steps.

$$\lim_{x\to -3} \frac{\sqrt{x^2-5}}{4-2x} = \frac{\lim_{x\to -3}\sqrt{x^2-5}}{\lim_{x\to -3}(4-2x)} \quad \text{(Rule 5, table 1)}$$

$$= \frac{\sqrt{\lim_{x\to -3}(x^2-5)}}{\left(\lim_{x\to -3} 4 - \lim_{x\to -3} 2x\right)} \quad \begin{matrix}\text{(rule 6, table 2)} \\ \text{(rule 2, table 1)}\end{matrix}$$

$$= \frac{\sqrt{\lim_{x\to -3} x^2 - \lim_{x\to -3} 5}}{4 - 2\lim_{x\to -3} x} \quad \begin{matrix}\text{(rule 2, T1)} \\ \text{(rule 3, T1)}\end{matrix}$$

$$= \frac{\sqrt{(-3)^2-5}}{4-2(-3)} \quad \begin{matrix}\text{(rule 5, T2)} \\ \text{(rule 3, T2)}\end{matrix}$$

$$= \frac{\sqrt{9-5}}{4+6} = \frac{\sqrt{4}}{10} = \boxed{\frac{1}{5}}$$

PRACTICE PROBLEM: 1 Let $f(x) = \frac{x^2+1}{2x-4}$.

a. Find $\lim_{x\to -1} f(x) = \frac{\lim_{x\to -1}(x^2+1)}{\lim_{x\to -1}(2x-4)}$

$$= \frac{1+1}{-2-4} = \frac{2}{-6} = \boxed{-1/3}$$

these are the same.

b. Find $f(-1) = \frac{(-1)^2+1}{2(-1)-4} = \frac{2}{-6} = \boxed{-1/3}$

c. Find $\lim_{x\to 2^+} f(x)$.

note $\lim_{x\to 2^+}(2x-4) = 0$, so our rules do not apply.

As $x \to 2^+$, $2x-4 \to 0^+$

And $\lim_{x\to 2^+}(x^2+1) = 5$

so $\boxed{\lim_{x\to 2^+} f(x) = \infty}$

d. Find $f(2)$

$\boxed{f(2) \text{ does not exist}}$

division by zero!

e. **T or F**: $\lim_{x\to -1} f(x) = f(-1)$.

$\boxed{\text{True}}$

$\boxed{\text{both are } -1/3}$

f. **T or F**: $\lim_{x\to 2} f(x) = f(2)$.

$\boxed{\text{False}}$

neither of these exist, so they cannot be equal.

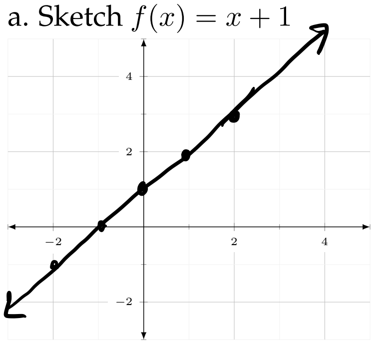
g. Fill in the blank in the statement of the DIRECT SUBSTITUTION PROPERTY:

If $f(x)$ is a polynomial or rational function and a is in the domain of f, then

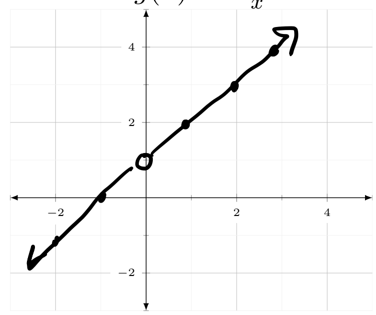
$\lim_{x\to a} f(x) =$ f(a) ← As in, you can just plug in a.

PRACTICE PROBLEM 2:

a. Sketch $f(x) = x+1$



b. Sketch $g(x) = \frac{x^2+x}{x} = \frac{x(x+1)}{x} = \boxed{x+1,\ x \neq 0}$



c. Find $f(0) = 0+1 = \boxed{1}$

d. Find $g(0)$. $\boxed{\text{undefined}}$

e. Find $\lim_{x\to 0} f(x) = \boxed{1}$

f. Find $\lim_{x\to 0} g(x) = \boxed{1}$

g. For what x-values is $f(x) = g(x)$? For what x-values is $f(x) \neq g(x)$?

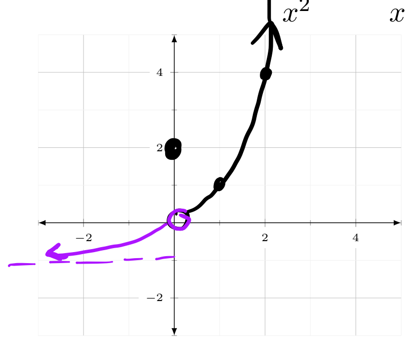
for x in $(-\infty, 0) \cup (0, \infty)$ } for $x=0$ only.

h. Explain how $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x)$ even though $f(0) \neq g(0)$.

when we find the limit as $x \to 0$ we only consider x close to 0, not $x=0$ itself.

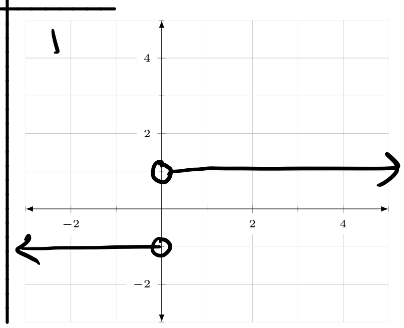
PRACTICE PROBLEM 3: Sketch the graph of each function below and find the indicated limits, if they exists. If the limits do not exist, explain why they do not exist.

a. Sketch $f(x) = \begin{cases} e^x - 1 & x < 0 \\ 2 & x = 0 \\ x^2 & x > 0 \end{cases}$



b. $g(x) = \frac{|x|}{x}$.

1	1



Table

x	
2	1
3	1
-1	-1
-2	-1
-3	-1

$\lim_{x\to -2} f(x) = \boxed{e^{-2}-1} = \boxed{1/e^2 - 1}$

$\lim_{x\to 3} g(x) = \boxed{1}$

$\lim_{x\to 0} f(x) = \boxed{0}$

$\lim_{x\to 0} g(x)$. $\boxed{\text{DNE}}$ as $\boxed{\lim_{x\to 0^-} g(x) \neq \lim_{x\to 0^+} g(x)}$

In general, describe the relationship between the TWO-sided limit and each of the ONE-sided limits.

$\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$

Q: what to do with limits that are 0/0 form?

A: use algebra to simplify so you can use direct substitution.

ex) $\lim_{x\to -4} \frac{x^2+5x+4}{x^2+3x-4} = \lim_{x\to -4} \frac{(x+4)(x+1)}{(x+4)(x-1)}$

$= \lim_{x\to -4} \left(\frac{x+1}{x-1}\right)$

$= \frac{-4+1}{-4-1}$

$= -3/-5$

$= \boxed{3/5}$

ex) $\lim_{x\to 16} \frac{4-\sqrt{x}}{16x-x^2} = \lim_{x\to 16} \frac{(4-\sqrt{x})(4+\sqrt{x})}{x(16-x)(4+\sqrt{x})}$

$= \lim_{x\to 16} \frac{16-x}{x(16-x)(4+\sqrt{x})}$

$= \lim_{x\to 16} \frac{1}{x(4+\sqrt{x})}$

$= \frac{1}{16(4+\sqrt{16})}$

$= \frac{1}{16(8)}$

$= \boxed{\frac{1}{128}}$

ex) $\lim_{h\to 0} \frac{(3+h)^{-1}-3^{-1}}{h} = \lim_{h\to 0} \frac{\frac{3}{3}\frac{1}{(3+h)} - \frac{1}{3}\left(\frac{3+h}{3+h}\right)}{h}$

$= \lim_{h\to 0} \frac{\frac{3}{3(3+h)} - \frac{(3+h)}{3(3+h)}}{h}$

$= \lim_{h\to 0} \frac{(3-3-h)}{3(3+h)} \cdot \frac{1}{h}$

$= \lim_{h\to 0} \frac{-h}{3(3+h)} \cdot \frac{1}{h}$

$= \lim_{h\to 0} \frac{-1}{3(3+h)}$

$= \boxed{-1/9}$