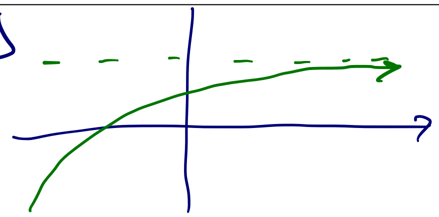
LECTURE: 2-6 LIMITS AT INFINITY [PART 1]

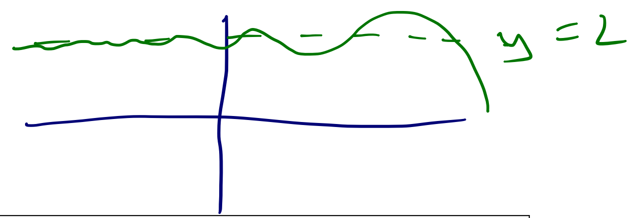
Intuitive Definition of a Limit at Infinity Let f be a function defined on some interval (a, ∞) or $(-\infty, a)$. Then

$$\lim_{x\to\infty} f(x) = 2 \quad \left(\text{or } \lim_{x\to-\infty} f(x) = L\right)$$

means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be big enough...

ex)



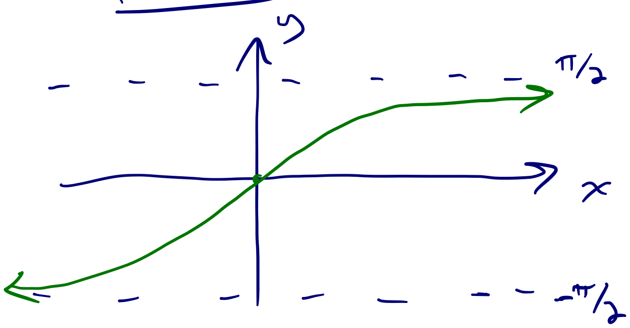


The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L$$

Example 1: Sketch a graph of $y = \tan^{-1} x$ and find the $\lim_{x\to\infty} \tan^{-1} x$ and $\lim_{x\to-\infty} \tan^{-1} x$.

Recall:

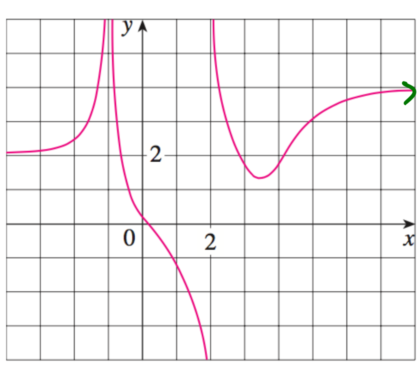


Thus

- $\lim_{x\to\infty} \tan^{-1} x = \pi/2$
- $\lim_{x\to-\infty} \tan^{-1} x = -\pi/2$

2.2 $\qquad x \to a$

Example 2: Find the infinite limits, limits at infinity, and asymptotes for the function f whose graph is shown below.



Infinite limits:
- $\lim_{x\to-1} f(x) = \infty$
- $\lim_{x\to2^-} f(x) = -\infty$
- $\lim_{x\to2^+} f(x) = \infty$

limits at infinity
- $\lim_{x\to\infty} f(x) = 4$
- $\lim_{x\to-\infty} f(x) = 2$

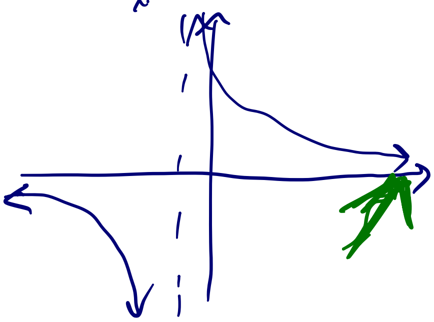
So asymptotes:
vertical: $x = -1$, $x = 2$
horizontal: $y = 2$, $y = 4$

Example 2: Find the following limits.

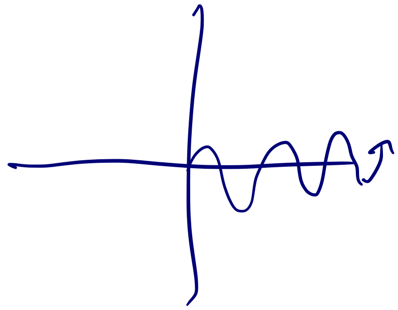
a) $\lim\limits_{x\to\infty} \dfrac{1}{7x+1} = 0$

think: $1 \div (\text{big } \#)$

$x=-1/7 \to 0$



b) $\lim\limits_{x\to\infty} \sin x = \text{DNE}$



c) $\lim\limits_{x\to\infty} 3e^{-x} = \lim\limits_{x\to\infty} \dfrac{3}{e^{x}} = 0$

$\therefore 3 \div (\text{monster } \#)$

$\to 0$

Big idea: Often, terms in function we are interested in finding the limit of will reduce to "$\frac{k}{0}$" or "$\frac{k}{\infty}$" when we attempt to evaluate them. We can think of these terms as going to...

"$\dfrac{k}{0}$" $\to \infty$ "$\dfrac{k}{\infty}$" $\to 0$

2.2   2.6

How to Determine Limits at Infinity: Divide the numerator and denominator by the highest common power between the numerator and denominator.

Example 3: Find the limit.

(a) $\lim\limits_{x\to\infty} \dfrac{(2x+5)\cdot 1/x}{(x-4)\cdot 1/x}$

$= \lim\limits_{x\to\infty} \dfrac{2 + 5/x}{1 - 4/x}$

$= \dfrac{2+0}{1-0} = \boxed{2}$

(b) $\lim\limits_{x\to\infty} \dfrac{(x+4)\cdot 1/x}{(x^2+x-3)\cdot 1/x}$

$= \lim\limits_{x\to\infty} \dfrac{1 + \cancel{4/x}^{\to 0}}{x + 1 - \cancel{3/x}^{\to 0}} = \boxed{0}$

$1 \div (\text{big}\#)$

Example 4: Evaluate the following limits.

(a) $\lim_{x\to\infty} \frac{2x^2+5}{3x^2+1}$ $\cdot \frac{1}{x^2}$ / $\cdot \frac{1}{x^2}$

$= \lim_{x\to\infty} \frac{2+5/x^2}{3+1/x^2}$

$= \frac{2+0}{3+0} = \boxed{\frac{2}{3}}$

(b) $\lim_{x\to\infty} \frac{2x+5}{3x^2+1}$ $\cdot \frac{1}{x}$ / $\cdot \frac{1}{x}$

$= \lim_{x\to\infty} \frac{2+\cancel{5/x}^{\to 0}}{3x+\cancel{1/x}^{\to 0}}$

$\boxed{=0}$

(c) $\lim_{x\to\infty} \frac{2x^3+5}{3x^2+1}$ $\cdot \frac{1}{x^2}$ / $\cdot \frac{1}{x^2}$

$= \lim_{x\to\infty} \frac{2x+5/x^2}{3+1/x^2}$

$= \boxed{\infty}$

(big #) $\cdot 2 \div 3 \to \infty$

Example 5: Find the following limits at infinity.

(a) $\lim_{x\to\infty} \frac{1+5e^x}{7-e^x}$ $\cdot \frac{1}{e^x}$ / $\cdot \frac{1}{e^x}$

$= \lim_{x\to\infty} \frac{1/e^x+5}{7/e^x-1}$

$= \boxed{-5}$

(b) $\lim_{x\to\infty} [\ln(2+x) - \ln(1+x)]$

$= \lim_{x\to\infty} \ln\left(\frac{2+x}{1+x}\right)$

$= \ln\left(\lim_{x\to\infty} \frac{2+x}{1+x} \begin{matrix}\cdot 1/x \\ \cdot 1/x\end{matrix}\right)$

$= \ln\left(\lim_{x\to\infty} \frac{2/x+1}{1/x+1}\right)$

$= \ln(1) = \boxed{0}$

Example 6: Find the limit.

$* \; 1/x = \frac{1}{\sqrt{x^2}}$

(a) $\lim_{x\to\infty} \frac{x+2}{\sqrt{9x^2+1}}$ $\cdot \frac{1}{x}$ / $\cdot \frac{1}{x}$

$= \lim_{x\to\infty} \frac{1+2/x}{\sqrt{(9x^2+1)\cdot 1/x^2}}$

$= \lim_{x\to\infty} \frac{1+2/x}{\sqrt{9+1/x^2}}$

$= \frac{1}{\sqrt{9}} = \boxed{1/3}$

(b) $\lim_{x\to\infty} \frac{\sqrt{3x^6-x}}{x^3+1}$ $\cdot \frac{1}{x^3}$ / $\cdot \frac{1}{x^3}$

$\frac{1}{x^3} = \frac{1}{\sqrt{x^6}}$

$= \lim_{x\to\infty} \frac{\sqrt{(3x^6-x)(1/x^6)}}{1+1/x^3}$

$= \lim_{x\to\infty} \frac{\sqrt{3-1/x^5}}{1+1/x^3} = \boxed{\sqrt{3}}$