LECTURE: 2-6 LIMITS AT INFINITY

Intuitive Definition of a Limit at Infinity Let f be a function defined on some interval (a, ∞) or $(-\infty, a)$. Then

$$\lim_{x\to\infty} f(x) = L \quad (\text{or } \lim_{x\to-\infty} f(x) = L)$$

means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be big enough or

How do deal with limits as $x \to -\infty$:

Replace x by $-x$. Then take limit as $x \to \infty$

Example 7: Find the limit.

(a) $\displaystyle \lim_{x\to-\infty} \frac{2x}{\sqrt{x^2+2}} = \lim_{x\to\infty} \frac{2(-x)}{\sqrt{(-x)^2+2}}$

$$= \lim_{x\to\infty} \frac{-2x}{\sqrt{x^2+2}} \cdot \frac{1/x}{1/x}$$

$$= \lim_{x\to\infty} \frac{-2}{\sqrt{(x^2+2)(1/x^2)}}$$

$$= \lim_{x\to\infty} \frac{-2}{\sqrt{1+2/x^2}} = \frac{-2}{\sqrt{1+0}} \boxed{=-2}$$

(b) $\displaystyle \lim_{x\to-\infty} (5 - 3e^x) = \lim_{x\to\infty} (5 - 3e^{-x})$

$$= \lim_{x\to\infty} \left(5 - \frac{3}{e^x}\right)$$

$$= 5 - 0 = \boxed{5}$$

Example 8: Evaluate the following limits.

(a) $\displaystyle \lim_{x\to\infty} (\sqrt{x^4+6x^2} - x^2) \cdot \frac{(\sqrt{x^4+6x^2}+x^2)}{(\sqrt{x^4+6x^2}+x^2)}$

$$= \lim_{x\to\infty} \frac{x^4+6x^2-x^4}{\sqrt{x^4+6x^2}+x^2}$$

$$= \lim_{x\to\infty} \frac{6x^2}{\sqrt{x^4+6x^2}+x^2} \frac{(1/x^2)}{(1/x^2)}$$

$$= \lim_{x\to\infty} \frac{6}{\sqrt{1+6/x^2}+1} = \frac{6}{1+1}$$

$$= \boxed{3}$$

(b) $\displaystyle \lim_{x\to\infty} (\sqrt{x^2+1} - x) \frac{(\sqrt{x^2+1}+x)}{(\sqrt{x^2+1}+x)}$

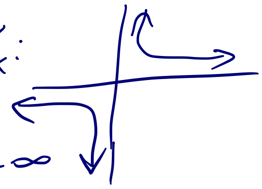
$$= \lim_{x\to\infty} \frac{(x^2+1)-x^2}{\sqrt{x^2+1}+x}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{x^2+1}+x} = \boxed{0}$$

Example 9: Evaluate the following limits.

(a) $\lim\limits_{x\to 0^-} e^{1/x}$

$\frac{1}{x}$:



as $x \to 0^-$, $\frac{1}{x} \to -\infty$

$\lim\limits_{x\to 0^-} e^{1/x} = \lim\limits_{x\to -\infty} e^{x} \boxed{= 0}$

$= \left(e^{-\text{big}\#}\right)$

Squeez!!

(b) $\lim\limits_{x\to\infty} e^{-2x}\cos x = \lim\limits_{x\to\infty} \dfrac{\cos x}{e^{2x}}$

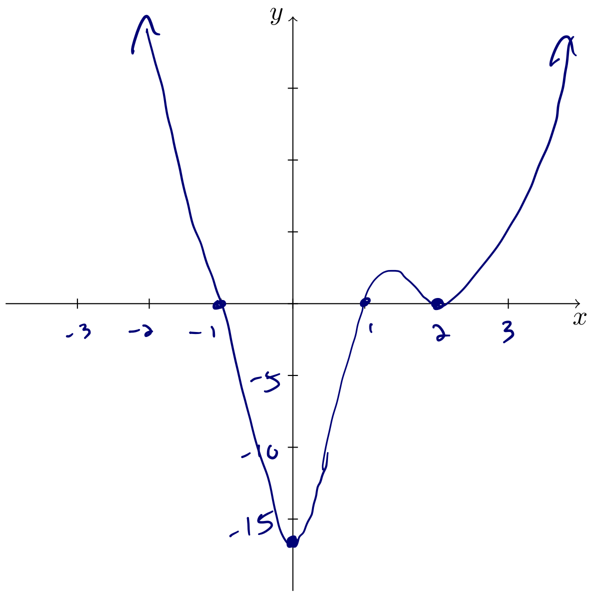
note: $-1 \le \cos x \le 1$

so $-1/e^{2x} \le \dfrac{\cos x}{e^{2x}} \le \dfrac{1}{e^{2x}}$

Since $\lim\limits_{x\to\infty} -1/e^{2x} = \lim\limits_{x\to\infty} \dfrac{1}{e^{2x}} = 0$,

$\lim\limits_{x\to\infty} \dfrac{\cos x}{e^{2x}} = 0$ by squeeze theorem!

Example 11: Sketch the graph of $y = (x-2)^4(x+1)^3(x-1)$ by finding its intercepts and its limits as $x \to \pm\infty$.



intercepts: $x = 0$:

y int: $y = (0-2)^4(0+1)^3(0-1) = -16$

x-int: $y = 0$

$(x-2)^4(x+1)^3(x-1) = 0$

$x = 2, -1, 1$

as $x \to \infty$, $y \to \infty$

as $x \to -\infty$,

$y \to (\text{big pos})(\text{big neg})(\text{big neg})$

$\Rightarrow y \to \infty$

Example 12: Find the horizontal and vertical asymptotes of $f(x) = \dfrac{\sqrt{16x^2+1}}{2x-8}$.

Vert. asymp:

$2x - 8 = 0$

$\boxed{x = 4}$

HA

$\lim\limits_{x\to\infty} \dfrac{\sqrt{16x^2+1}}{2x-8} \cdot \dfrac{1/x}{1/x}$

$= \lim\limits_{x\to\infty} \dfrac{\sqrt{16 + 1/x^2}}{2 - 8/x} = \dfrac{\sqrt{16}}{2}$

$= 2$

$\boxed{y = 2}$

$\lim\limits_{x\to -\infty} \dfrac{\sqrt{16x^2+1}}{2x-8}$

$= \lim\limits_{x\to\infty} \dfrac{\sqrt{16x^2+1}}{-2x-8} \cdot \dfrac{1/x}{1/x}$

$= \lim\limits_{x\to\infty} \dfrac{\sqrt{16 + 1/x^2}}{-2 - 8/x}$

$= \dfrac{\sqrt{16}}{-2} = -2$

$\boxed{y = -2}$