LECTURE: 3-3 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

Recall last time we found $\frac{d}{dx}(\sin x) = \cos x$ and $\frac{d}{dx}(\cos x)$.

Example 1: Using the derivative of $\sin x$ and $\cos x$ find derivatives of:

(a) $y = \cot x$ $= \dfrac{\cos x}{\sin x}$

$$y' = \frac{\sin x(\cos x)' - \cos x(\sin x)'}{(\sin x)^2}$$

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$$

$$= -\frac{1}{\sin^2 x} = \boxed{-\csc^2 x}$$

(b) $y = \csc x$ $= \dfrac{1}{\sin x}$

$$y' = \frac{\sin x\,\underbrace{(1)'}_{0} - 1(\sin x)'}{(\sin x)^2}$$

$$= \frac{-\cos x}{\sin^2 x} = \boxed{-\cot x \csc x}$$

Derivatives of Trigonometric Functions:

- $\frac{d}{dx}(\sin x) =$ $\cos x$
- $\frac{d}{dx}(\cos x) =$ $-\sin x$
- $\frac{d}{dx}(\tan x) =$ $\sec^2 x$
- $\frac{d}{dx}(\csc x) =$ $-\cot x \csc x$
- $\frac{d}{dx}(\sec x) =$ $\sec x \tan x$
- $\frac{d}{dx}(\cot x) =$ $-\csc^2 x$

Example 2: Find the ~~second~~ first derivatives of the following functions:

(a) $g(t) = 4\sec t + \tan t$.

$$g'(t) = 4\sec t\tan t + \sec^2 t$$

$$\boxed{= \sec t(4\tan t + \sec t)}$$

(b) $y = x^2 \sin x$.

$$y' = 2x\sin x + x^2\cos x$$

$$\boxed{= x(2\sin x + x\cos x)}$$

Example 3: Find an equation of the tangent line to the curve $y = \dfrac{1}{\sin x + \cos x}$ at the point $(0, 1)$.

$$y' = \frac{(\sin x + \cos x)(0) - 1(\cos x - \sin x)}{(\sin x + \cos x)^2}$$

$$= \frac{\sin x - \cos x}{(\sin x + \cos x)^2}$$

$x=0$: $y'(0) = \dfrac{\sin 0 - \cos 0}{(\sin 0 + \cos 0)^2} = \dfrac{-1}{1^2} = -1$

$$y - 1 = -1(x - 0)$$

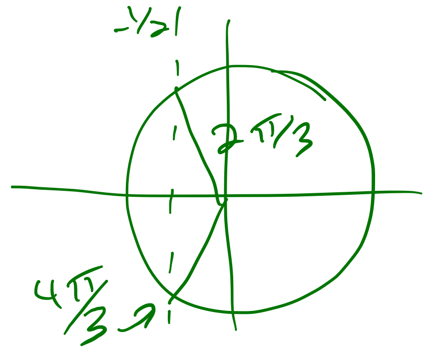
$$\boxed{y = -x + 1}$$

Example 4: For what values of x does the graph of $f(x) = x + 2\sin x$ have a horizontal tangent?

$f'(x) = 1 + 2\cos x = 0$

$\cos x = -\frac{1}{2}$

$$\boxed{x = \frac{2\pi}{3} + 2k\pi \text{ or } \frac{4\pi}{3} + 2k\pi}$$



Example 5: Differentiate $f(x) = \dfrac{\sec x}{1 - \tan x}$ and determine where the tangent line is horizontal.

$$f'(x) = \frac{(1-\tan x)(\sec x \tan x) - \sec x(-\sec^2 x)}{(1-\tan x)^2}$$

$c^2 + s^2 = 1$

$1 + \tan^2 = \sec^2$

$$= \frac{\sec x(\tan x - \tan^2 x + \sec^2 x)}{(1-\tan x)^2}$$

$$= \frac{\sec x(\tan x + 1)}{(1-\tan x)^2}$$

$\sec x = 0$ ✗ – no values

$\tan x + 1 = 0$

$\tan x = -1$ $x = 3\pi/4$

$$\boxed{x = \frac{3\pi}{4} + k\pi}$$

Generalized Product Rule: How does the product rule genearlize to more than two functions? For example, what is the derivative of $y = f(x)g(x)h(x)$?

$f(x) \cdot (g(x)h(x))$

power rule within power rule!

$y' = f(x)(g(x)\cdot h'(x) + g'(x)h(x)) + f'(x)(g(x)h(x))$

Example 6: Differentiate $h(\theta) = \theta^2(\tan\theta\sec\theta.)$

$h'(\theta) = \theta^2(\tan\theta\sec\theta)' + 2\theta\tan\theta\sec\theta$

$= \theta^2(\tan\theta \cdot \sec\theta\tan\theta + \sec^2\theta\sec\theta) + 2\theta\tan\theta\sec\theta$

$$\boxed{= \theta^2(\sec\theta\tan^2\theta + \sec^3\theta) + 2\theta\tan\theta\sec\theta}$$

or

$$\boxed{\theta\sec\theta(\theta\tan^2\theta + \theta\sec^2\theta + 2\tan\theta)}$$

Example 7: Find the 51st derivative of $f(x) = \sin x$. Specifically, find the first four or five derivatives and look for a pattern.

$f'(x) = \cos(x)$

$f''(x) = -\sin x$

$f'''(x) = -\cos x$

$f^{(4)}(x) = \sin x$

hmm. 51 is odd. Either $\cos x$ or $-\cos x$

52 is divisible by 4, so is 4

so $f^{(51)}(x) = f'''(x) = \boxed{-\cos x}$

Example 8: A mass on a spring vibrates horizontally on a smooth level surface. Its equation of motion is $x(t) = 8 \sin t$, where t is in seconds and x is in centimeters.

positive x, to the right

(a) Find the velocity at time t.

$x'(t) = 8 \cos t$

(b) Find the position and velocity of the mass at time $t = 2\pi/3$. In what direction is it moving at this time?

$x(2\pi/3)$ $\quad x'(2\pi/3)$

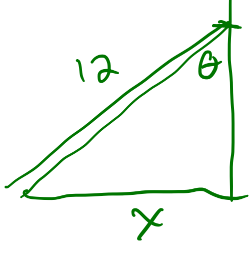
position: $x\left(\frac{2\pi}{3}\right) = 8 \sin\left(\frac{2\pi}{3}\right) = 8 \frac{\sqrt{3}}{2} = \boxed{4\sqrt{3} \text{ cm}}$

velocity: $x'\left(\frac{2\pi}{3}\right) = 8 \cos\left(\frac{2\pi}{3}\right) = 8\left(-\frac{1}{2}\right) = \boxed{-4 \text{ cm/sec}}$

Left

Example 9: A ladder 12 feet long rests against a vertical wall. Let θ be the angle between the top of the ladder and the wall and let x be the distance from the bottom of the ladder to the wall. If the bottom of the ladder slides away from the wall, how fast does x change with respect to θ when $\theta = \frac{\pi}{6}$.

$\frac{dx}{d\theta}$



$\sin\theta = \frac{x}{12}$

$\Rightarrow \quad x = 12 \sin\theta$

$\frac{dx}{d\theta} = \frac{d}{d\theta}(12 \sin\theta)$

$= 12\cos\theta$

$x'(\pi/6) = 12 \cos \pi/6 = 12\frac{\sqrt{3}}{2} = \boxed{6\sqrt{3} \text{ ft/radians}}$