LECTURE: 3-4 THE CHAIN RULE

When you have a function that is a composite function, like $y = \sqrt{x^2+1}$, the formulas we have so far do not let us find y'. However, if you write your composite function as $f \circ g$, we have a formula for the derivative.

The Chain Rule: If f and g are differentiable and $F = f \circ g$, then F is differentiable and

$$F'(x) = \underline{f'(g(x))g'(x)}.$$

$$\frac{d}{dx}\left(f(\text{blah})\right) = f'(\text{blah}) \cdot \text{blah}'$$

$f(x) = x^{-9}$
$g(x) = \dots$

Example 1: Write the composite function in the form $f(g(x))$ and then find y'.

(a) $y = (1+3x)^9$

$f(x) = x^9$
$g(x) = 1+3x$

$$y' = 9(1+3x)^8 \frac{d}{dx}(1+3x)$$

$$\boxed{= 27(1+3x)^8}$$

(b) $y = \dfrac{1}{(x^2+2x-5)^9}$ $= (x^2+2x-5)^{-9}$

$$y' = -9(x^2+2x-5)^{-10} \frac{d}{dx}(x^2+2x-5)$$

$$\boxed{= \frac{-9(2x+2)}{(x^2+2x-5)^{10}}}$$

Example 2: Write the composite function in the form $f(g(x))$. Then, find y'.

(a) $y = \cos(x^3)$

$f(x) = \cos x$
$g(x) = x^3$

$$y' = -\sin(x^3) \cdot \frac{d}{dx}(x^3)$$

$$\boxed{= -3x^2 \sin(x^3)}$$

(b) $y = \cos^3(x)$ $= (\cos x)^3$

$f(x) = x^3$
$g(x) = \cos x$

$$y' = 3\cos^2 x \cdot \frac{d}{dx}(\cos x)$$

$$\boxed{= -3\cos^2 x \sin x}$$

Example 3: Find the derivative of $f(x) = (2x-1)^6(x^3-2x+1)^3$ ⇐ a product!

$$f'(x) = \frac{d}{dx}\left((2x-1)^6\right) \cdot (x^3-2x+1)^3 + (2x-1)^6 \cdot \frac{d}{dx}\left[(x^3-2x+1)^3\right]$$

$$= 6(2x-1)^5(2)(x^3-2x+1)^3 + (2x-1)^6 \cdot 3(x^3-2x+1)^2(3x^2-2)$$

$$\boxed{= 3(2x-1)^5(x^3-2x+1)^2\left[4(x^3-2x+1) + (2x-1)(3x^2-2)\right]}$$

$\leftarrow$ a power! chain first.

Example 4: Find the derivative of $f(x) = \left(\frac{x+5}{2x-1}\right)^5$.

$$f'(x) = 5\left(\frac{x+5}{2x-1}\right)^4 \cdot \frac{d}{dx}\left(\frac{x+5}{2x-1}\right)$$

$$= 5\left(\frac{x+5}{2x-1}\right)^4 \cdot \left(\frac{(2x-1)-2(x+5)}{(2x-1)^2}\right)$$

$$= 5\left(\frac{x+5}{2x-1}\right)^4 \left(\frac{-11}{(2x-1)^2}\right)$$

$$\boxed{= \frac{-55(x+5)^4}{(2x-1)^6}}$$

Example 5: Find the derivative of the following functions.

chain rule twice!

(a) $y = e^{\sec x}$

$$y' = e^{\sec x} \cdot \frac{d}{dx}(\sec x)$$

$$\boxed{= e^{\sec x} \cdot \sec x \tan x}$$

(b) $y = \sin(\sin(\sin x))$

$$y' = \cos(\sin(\sin x)) \cdot \frac{d}{dx}(\sin(\sin x))$$

$$= \cos(\sin(\sin x)) \cdot \cos(\sin x) \cdot \frac{d}{dx}\sin x$$

$$\boxed{= \cos(\sin(\sin x)) \cdot \cos(\sin x) \cdot \cos x}$$

Example 6: Let $F(x) = f(g(x))$, where $f(-2) = 8$, $f'(-2) = 4$, $f'(5) = 3$, $g(5) = -2$, and $g'(5) = 6$, find $F'(5)$.

$$F'(x) = f'(g(x)) \cdot g'(x)$$

$$F'(5) = f'(-2) \cdot (6) = 4(6) = \boxed{24}$$

Example 7: Find the derivative of the following functions.

(a) $g(x) = \sqrt[5]{x^3 - 1}$ $= (x^3-1)^{1/5}$

$$g'(x) = \frac{1}{5}(x^3-1)^{-4/5} \frac{d}{dx}(x^3)$$

$$\boxed{= \frac{3x^2}{5(x^3-1)^{4/5}}}$$

(b) $h(x) = \sin^5(4x^2)$

$$h'(x) = 5\sin^4(4x^2) \cdot \frac{d}{dx}(\sin(4x^2))$$

$$= 5\sin^4(4x^2) \cdot \cos(4x^2) \frac{d}{dx}(4x^2)$$

$$\boxed{= 40x \sin^4(4x^2) \cdot \cos(4x^2)}$$

Formula: Derivative of $y = b^x$:

$$\frac{d}{dx}(b^x) = (\ln b)\cdot b^x$$

$y = b^x = (e^{\ln b})^x = e^{\ln b \cdot x}$

so, $y' = e^{\ln b\cdot x} \cdot \frac{d}{dx}(\ln b \cdot x)$

$= b^x \cdot \ln b$

Note: if $b = e$,

$y = e^x$

$\Rightarrow y' = (\ln e) e^x = 1\, e^x$ ✓

Example 8: Find the derivative of the following functions.

(a) $y = 5^x$

$y' = (\ln 5)\cdot 5^x$

(b) $f(x) = 10^{\cos x}$

$f'(x) = (\ln 10)\, 10^{\cos x} \cdot \frac{d}{dx}(\cos x)$

$\boxed{= (-\ln 10)\, 10^{\cos x} \sin x}$

(c) $g(x) = e^{-2x^2}$

$g'(x) = e^{-2x^2} \cdot \frac{d}{dx}(-2x^2)$

$\boxed{= -4x e^{-2x^2}}$

Example 9: Find the derivative of the following functions.

(a) $f(x) = 5^{3^{x^2}}$

$f'(x) = 5^{3^{x^2}} \cdot \ln(5) \cdot \frac{d}{dx}(3^{x^2})$

$= \ln(5)\ln(3) \cdot 5^{3^{x^2}} \cdot 3^{x^2} \frac{d}{dx}(x^2)$

$\boxed{= 2\ln(5)\ln(3)\, x\, 5^{3^{x^2}} \cdot 3^{x^2}}$

(b) $y = \sin\sqrt{\cancel{\cos}(3x)}$

$y = \sin(3x)^{1/2}$

$y' = \cos(3x)^{1/2} \cdot \frac{d}{dx}(3x)^{1/2}$

$= \cos(3x)^{1/2} \cdot \frac{1}{2}(3x)^{-1/2} \cdot 3$

$= \boxed{\frac{3\cos\sqrt{3x}}{2\sqrt{3x}}}$

Example 10: Find the points on the graph of the function $f(x) = 2\cos x + \cos^2 x$ at which the tangent is horizontal.

$f'(x) = -2\sin x + 2\cos x(-\sin x) = 0$

$-2\sin x(1 + \cos x) = 0$

$\sin x = 0$

$\boxed{x = k\pi}$

$\cos x = -1$

$x = \pi + 2k\pi$

$\boxed{x = k\pi}$

Example 11: Find the 100th derivative of $y = \sin(5x)$.

$y' = 5\cos(5x)$

$y'' = -25\sin(5x)$

$y''' = -(5)^3\cos(5x)$

$y^{(4)} = (5)^3\sin(5x)\cdot 5 = 5^4\sin(5x)$

$y^{(100)} = 5^{100}\sin(5x)$

Example 12: A model for the length of day (in hours) in Philadelphia on the t-th day of the year is

$$L(t) = 12 + 2.8\sin\left[\frac{2\pi}{365}(t-80)\right].$$

Use this model to compare the number of hours of daylight is increasing in Philadelphia on January 15th ($t = 15$) and March 21st ($t = 80$).

$L'(t) = 2.8\cos\left(\frac{2\pi}{365}(t-80)\right)\cdot\frac{d}{dt}\left(\frac{2\pi}{365}(t-80)\right)$

$= 2.8\cdot\frac{2\pi}{365}\cos\left(\frac{2\pi}{365}(t-80)\right)$

$L'(15) \approx 0.021$ hrs/dy   1.26 min/day

$L'(80) = 2.8\cdot\frac{2\pi}{365}(\cos 0) \approx 0.048$ hrs/day

2.89 min/day

Example 13: Use the product rule and chain rule to prove the quotient rule.

$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{d}{dx}\left(f\cdot g^{-1}\right)$

$= f'\cdot g^{-1} + f\cdot(-1)g^{-2}\cdot g'$ chain rule!

$= \frac{f'}{g} - \frac{fg'}{g^2} = \boxed{\frac{gf' - fg'}{g^2}}$

Example 14: Find the derivatives of the following functions.

(a) $y = \cos^2(\cot(2x))$

$y' = 2\cos(\cot(2x)) \cdot \frac{d}{dx}(\cot(2x))$

$= 2\cos(\cot(2x)) \cdot (-\csc(2x)\cot(2x))(2)$

$\boxed{= -4\cos(\cot(2x))\csc(2x)\cot(2x)}$

(b) $y = x^3 e^{-1/x^2}$

product!

$y' = x^3 \frac{d}{dx}(e^{-1/x^2}) + 3x^2 e^{-1/x^2}$

$= x^3 e^{-1/x^2} \cdot \frac{d}{dx}(-\frac{1}{x^2}) + 3x^2 e^{-1/x^2}$

$= \frac{2}{x^3} \cdot x^3 e^{-1/x^2} + 3x^2 e^{-1/x^2}$

$\boxed{= 2e^{-1/x^2} + 3x^2 e^{-1/x^2}}$

Example 15: Find an equation of the tangent line to the curve $y = 3^{\sin x}$ at the point where $x = 0$.

$y = 3^0 = 1$

$y' = 3^{\sin x}(\ln 3) \cdot \frac{d}{dx}(\sin x)$

$= (\ln 3)\, 3^{\sin x} \cdot \cos x$

$y'\Big|_{x=0} = (\ln 3)\, 3^{\sin 0} \cdot \cos 0$

$= \ln 3 = m$

$y - 1 = \ln 3 (x - 0)$

$\boxed{y = (\ln 3)x + 1}$