Lecture: 3-5 Implicit Differentiation (Part 1)

So far, the functions that we have encountered can be described by expressing one variable as a function of the other. For example:

$y = x^2$, $y = \sin(2x) + \cos^2(2x)$

$y = \sqrt{x}$,

However, sometimes we have a relationship between x and y that is described implicitly. For example:

$x^2 + y^2 = 25$ (circle) $\qquad y^2 = 25 - x^2$

$x^3 + xy + y^2 = 1 \qquad y = \pm\sqrt{25 - x^2}$

While one could take the derivative of the expression above by solving for y and then taking the derivative, sometimes it is not easy (or even possible) to solve for y by hand. In these cases we may be still be interested in the derivative and the information it gives us about the curve. It turns out that one doesn't need to solve an equation for y to find the derivative. We simply need to use the method of **implicit differentiation** which is an application of the chain rule.

Example 1: Suppose we have a curve $y = g(x)$. How do you differentiate $[g(x)]^2$?

$$\frac{d}{dx}[g(x)]^2 = 2[g(x)]^1 \cdot g'(x) = 2\, g(x)\, g'(x)$$

OR $$\frac{d}{dx}[y]^2 = 2y \cdot y' \text{ or } 2y\frac{dy}{dx}$$

> **Implicit differentiation** is a method of finding the slope of the tangent line to curves where y is not necessarily a function of x. In other words, these equations cannot be easily solved for y. If you are finding $\frac{d}{dx}$, or the derivative with respect to the variable x, whenever you differentiate a term involving a y, you must add a $\frac{dy}{dx}$ or y'.

Example 2: If $x^2 + y^2 = 25$ find $\dfrac{dy}{dx}$. Then, find an equation of the tangent line to the circle at the point $(3, 4)$

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$$

$$2x\frac{dx}{dx} + 2y\frac{dy}{dx} = 0$$

$$2x + 2y\frac{dy}{dx} = 0$$

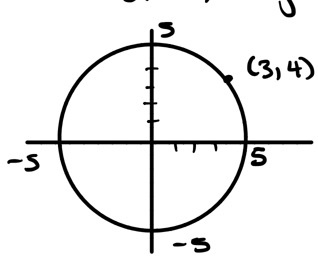
$$2y\frac{dy}{dx} = -2x$$

$$\boxed{\frac{dy}{dx} = \frac{-x}{y}}$$

$m = \frac{-3}{4}$; $y - y_1 = m(x - x_1)$

$$\boxed{y - 4 = -\frac{3}{4}(x - 3)}$$

check $\frac{dy}{dx}$ against a graph of $x^2+y^2 = 25$



(3,4) ← slope is negative!

observe, slope is zero if $x=0$

slope is ∞ if $y=0$,

which works w/ dy/dx

Example 3: Find $\frac{dy}{dx}$ by implicit differentiation.

(a) $x^3 + y^3 = 6xy$

must use the product rule here!

$$\frac{d}{dx}(x^3+y^3) = \frac{d}{dx}\, 6xy$$

$$3x^2 + 3y^2 \cdot \frac{dy}{dx} = 6y + 6x \cdot 1 \cdot \frac{dy}{dx}$$

$$3y^2 \frac{dy}{dx} - 6x\frac{dy}{dx} = 6y - 3x^2$$

$$\frac{dy}{dx}(3y^2 - 6x) = 6y - 3x^2$$

$$\frac{dy}{dx} = \frac{6y - 3x^2}{3y^2 - 6x} = \boxed{\frac{2y - x^2}{y^2 - 2x}}$$

(b) $\frac{2}{x} - \frac{1}{y} = 4$

$$\frac{d}{dx}(2x^{-1} - y^{-1}) = \frac{d}{dx}4$$

$$-2x^{-2} + y^{-2}\frac{dy}{dx} = 0$$

$$\frac{1}{y^2}\frac{dy}{dx} = \frac{2}{x^2}$$

$$\boxed{\frac{dy}{dx} = \frac{2y^2}{x^2}}$$

Don't forget that the product and quotient rule also apply when finding a derivative using implicit differentiation.

Example 4: Find $\frac{dy}{dx}$ by implicit differentiation.

(a) $2x^2 + xy - y^2 = 2$

$$\frac{d}{dx}(2x^2 + xy - y^2) = \frac{d}{dx}2$$

$$4x + 1y + x \cdot \frac{dy}{dx} - 2y\frac{dy}{dx} = 0$$

$$x\frac{dy}{dx} - 2y\frac{dy}{dx} = -4x - y$$

$$\frac{dy}{dx}(x - 2y) = -4x - y$$

$$\boxed{\frac{dy}{dx} = \frac{-4x - y}{x - 2y} = \frac{4x + y}{2y - x}}$$

(b) $y \cos x = x^2 + y^2$

$$\frac{d}{dx}(y\cos x) = \frac{d}{dx}(x^2 + y^2)$$

$$\frac{dy}{dx}\cos x + y(-\sin x) = 2x + 2y\frac{dy}{dx}$$

$$\frac{dy}{dx}\cos x - 2y\frac{dy}{dx} = 2x + (\sin x)y$$

$$\frac{dy}{dx}(\cos x - 2y) = 2y + y\sin x$$

$$\boxed{\frac{dy}{dx} = \frac{2y + y\sin x}{\cos x - 2y}}$$

Also, don't forget that sometimes the chain rule may appear along with the product, quotient or another application of the chain rule.

Example 5: Find $\frac{dy}{dx}$ by implicit differentiation.

(a) $e^{xy} = x + y$

$$\frac{d}{dx} e^{xy} = \frac{d}{dx}(x+y)$$

$$e^{xy}\frac{d}{dx}xy = 1 + \frac{dy}{dx}$$

$$e^{xy}\left(1y + x\frac{dy}{dx}\right) = 1 + \frac{dy}{dx}$$

$$ye^{xy} + xe^{xy}\frac{dy}{dx} = 1 + \frac{dy}{dx}$$

$$xe^{xy}\frac{dy}{dx} - \frac{dy}{dx} = 1 - ye^{xy}$$

$$\frac{dy}{dx}(xe^{xy} - 1) = 1 - ye^{xy}$$

$$\boxed{\frac{dy}{dx} = \frac{1-ye^{xy}}{xe^{xy}-1}}$$

(b) $\cos(xy) = 1 + \sin y$

$$\frac{d}{dx}\cos(xy) = \frac{d}{dx}(1+\sin y)$$

$$-\sin(xy)\frac{d}{dx}xy = 0 + \cos y \cdot \frac{dy}{dx}$$

$$-\sin(xy)\left(1y + x\frac{dy}{dx}\right) = \cos y\frac{dy}{dx}$$

$$-y\sin(xy) - x\sin(xy)\frac{dy}{dx} = \cos y\frac{dy}{dx}$$

$$-y\sin(xy) = x\sin(xy)\frac{dy}{dx} + \cos y\frac{dy}{dx}$$

$$-y\sin(xy) = \frac{dy}{dx}(x\sin(xy) + \cos y)$$

$$\boxed{\frac{dy}{dx} = \frac{-y\sin(xy)}{x\sin(xy)+\cos y}}$$

As long as you can compute the derivative you can also find equations of tangent lines for these curves just as you could for functions.

Example 6: Find anequation of the tangent line to $2(x^2 + y^2)^2 = 25(x^2 - y^2)$ at the point $(3, 1)$.

$$\frac{d}{dx}2(x^2+y^2)^2 = \frac{d}{dx}25(x^2-y^2)$$

$$2\cdot 2(x^2+y^2)\cdot\frac{d}{dx}(x^2+y^2) = 25\left(2x - 2y\frac{dy}{dx}\right)$$

$$4(x^2+y^2)\left(2x+2y\frac{dy}{dx}\right) = 25\left(2x - 2y\frac{dy}{dx}\right)$$

you don't have to solve for dy/dx, you can save time to input your point now!

$$4(3^2+1^2)(2\cdot 3 + 2\cdot 1\, m) = 25\left(2\cdot 3 - 2\cdot 1\frac{dy}{dx}\right)$$

$$4(10)(6+2m) = 25(6-2m) \quad (\div \text{ by } 10)$$

$$4\cdot 2\cdot(3+m) = 5(3-m)$$

$$24 + 8m = 15 - 5m$$

$$13m = -9$$

$$m = -9/13$$

$$y - y_1 = m(x - x_1)$$

$$\boxed{y - 1 = -9/13(x-3)}$$

We can also find the second derivative of a function that has been computed using implicit differentiation. After we complete example 7, note the tricks/ differences that you observe in finding y''.

Example 7: Given $x^2 + 9y^2 = 9$ find y'' by implicit differentiation.

$$2x + 18y\frac{dy}{dx} = 0$$

$$18y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{18y} = \frac{-x}{9y}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{-x}{9y}\right)$$

$$= \frac{9y(-1) - (-x)\,9\frac{dy}{dx}}{(9y)^2}$$

$$= \frac{-y + x\frac{dy}{dx}}{9y^2}$$

input $\frac{dy}{dx}$ here

$$\frac{d^2y}{dx^2} = \left(\frac{-y + x\left(\frac{-x}{9y}\right)}{9y^2}\right)\frac{9y}{9y}$$

$$= \frac{-9y^2 - x^2}{81y^3}$$

$$= \frac{-(x^2 + 9y^2)}{81y^3}$$

this is 9 from the original

$$= \frac{-9}{81y^3}$$

$$= \boxed{\frac{-1}{9y^3}}$$

Example 8: Given $\sin y + \cos x = 1$ find y'' by implicit differentiation.

$$\cos y\frac{dy}{dx} - \sin x = 0$$

$$\frac{dy}{dx} = \frac{\sin x}{\cos y}$$

$$\frac{d^2y}{dx^2} = \frac{\cos y \cdot \cos x - \sin x(-\sin y)\frac{dy}{dx}}{\cos^2 y}$$

$$= \left(\frac{\cos y\cos x + \sin x\sin y\left(\frac{\sin x}{\cos y}\right)}{\cos^2 y}\right)\frac{\left(\frac{\cos y}{1}\right)}{\left(\frac{\cos y}{1}\right)}$$

$$= \boxed{\frac{\cos^2 y\cos x + \sin^2 x\sin y}{\cos^3 y}}$$