LECTURE: 3-5 IMPLICIT DIFFERENTIATION (PART 2)

Example 1: Review. Find $\frac{dy}{dx}$ by implicit differentiation.

$\sin(x+y) - 2xy = 3$

$$\cos(x+y)\cdot\frac{d}{dx}(x+y) - 2\left(y + x\frac{dy}{dx}\right) = 0$$

$$\cos(x+y)\left(1+\frac{dy}{dx}\right) - 2y - 2x\frac{dy}{dx} = 0$$

$$\left[\cos(x+y) - 2x\right]\frac{dy}{dx} = 2y - \cos(x+y)$$

$$\boxed{\frac{dy}{dx} = \frac{2y - \cos(x+y)}{\cos(x+y) - 2x}}$$

Example 2: If $g(x) + x\sin g(x) = 3x^2 + 1$ and $g(1) = 0$ find $g'(1)$.

$(1, 0)$

$$g'(x) + \sin(g(x)) + x\cos(g(x))\, g'(x) = 6x$$

$$g'(1) + \sin(0) + 1\cos(0)\, g'(1) = 6$$

$$g'(1) + g'(1) = 6$$

$$2g'(1) = 6 \qquad \boxed{g'(1) = 3}$$

Derivatives of Inverse Trigonometric Functions

Implicit differentiation is also used to derive formulas for derivatives of inverse functions.

Example 3: Find the derivatives of the following functions.

(a) $y = \sin^{-1} x$

$\sin y = x$

$\frac{d}{dx}$

$\cos y \cdot \frac{dy}{dx} = 1$

$\cos^2 y + \sin^2 y = 1$

$\cos y = \sqrt{1 - \sin^2 y}$

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \boxed{\frac{1}{\sqrt{1-x^2}}}$$

(b) $y = \tan^{-1} x$

$\tan y = x$

$\sec^2 y \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{\sec^2 y}$

$$\frac{dy}{dx} = \frac{1}{1+\tan^2 y} = \boxed{\frac{1}{1+x^2}}$$

Derivatives of Inverse Trigonometric Functions:

- $\frac{d}{dx}(\sin^{-1} x) =$ $1/\sqrt{1-x^2}$
- $\frac{d}{dx}(\cos^{-1} x) =$ $-1/\sqrt{1-x^2}$
- $\frac{d}{dx}(\tan^{-1} x) =$ $1/1+x^2$

Example 4: Differentiate the following functions.

(a) $y = \cos^{-1}(3x + 5)$

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-(3x+5)^2}} \cdot \frac{d}{dx}(3x+5)$$

$$= \frac{-3}{\sqrt{1-(3x+5)^2}}$$

(b) $y = \arctan 2x$

$$\frac{dy}{dx} = \frac{1}{1+(2x)^2} \cdot \frac{d}{dx}(2x)$$

$$\boxed{= \frac{2}{1+4x^2}}$$

Example 5: Differentiate the following functions.

(a) $f(t) = \arcsin(\sqrt{t})$

$$f'(t) = \frac{1}{\sqrt{1-(\sqrt{t})^2}} \cdot \frac{d}{dt}(\sqrt{t})$$

$$= \frac{1}{\sqrt{1-t}} \cdot \frac{1}{2}t^{-1/2}$$

$$\boxed{= \frac{1}{2\sqrt{t}\sqrt{1-t}}}$$

(b) $y = x \sin^{-1} x + \sqrt{1 - x^2}$

$$y' = \sin^{-1}x + x \cdot \frac{1}{\sqrt{1-x^2}} + \frac{1}{2}(1-x^2)^{-1/2}(-2x)$$

$$= \sin^{-1}x + \frac{x}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}}$$

$$\boxed{= \sin^{-1}x}$$