LECTURE: 3-6 DERIVATIVES OF LOGARITHMIC FUNCTIONS

Review: Derivatives of Exponential Functions:

- $\frac{d}{dx}e^x = $ e^x
- $\frac{d}{dx}a^x = $ $a^x(\ln a)$

Example 1: Find a formula for the derivatives of the following functions.

(a) $y = \ln x$ $\iff e^y = x$

differentiate!

$$e^y \cdot \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{e^y}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{x}}$$

(b) $y = \log_b x$ $\iff b^y = x$

$$b^y(\ln b)\cdot\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{(\ln b)b^y}$$

$$\frac{dy}{dx} = \frac{1}{(\ln b)x}$$

Derivatives of Logarithmic Functions:

- $\frac{d}{dx}\ln x = $ $1/x$
- $\frac{d}{dx}\log_b x = $ $\frac{1}{(\ln b)x}$

Example 2: Find derivatives of the following functions.

(a) $y = \ln(4x^2 + 5)$

$$y' = \frac{1}{4x^2+5}\cdot\frac{d}{dx}(4x^2+5)$$

$$= \boxed{\frac{8x}{4x^2+5}}$$

(b) $y = \ln(\tan x)$

$$y' = \frac{1}{\tan x}\cdot\frac{d}{dx}(\tan x)$$

$$= \frac{\sec^2 x}{\tan x}$$

Example 3: Find derivatives of the following functions.

(a) $f(x) = \log_{10}\sqrt{x}$

$$f'(x) = \frac{1}{(\ln 10)\sqrt{x}} \cdot \frac{d}{dx}(\sqrt{x})$$

$$= \frac{1}{(\ln 10)\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}$$

$$= \boxed{\frac{1}{2(\ln 10)x}}$$

$\boxed{\begin{array}{c} x>0 \\ (0,\infty) \end{array}}$

(b) $g(x) = \sqrt{5 + \ln x}$

$$g'(x) = \frac{1}{2}(5+\ln x)^{-1/2} \cdot \frac{d}{dx}(\ln x)$$

$$= \frac{1}{2x\sqrt{5+\ln x}}$$

$x > 0$

$5 + \ln x > 0$

$\ln x > -5$

$\boxed{x > e^{-5}}$

Example 4: Differentiate the following functions.

(a) $y = \ln|x|$.

$\boxed{y' = \frac{1}{x}}$!

$x>0$: $y = \ln x$

$y' = \boxed{\frac{1}{x}}$

$x<0$: $y = \ln(-x)$

$$y' = \frac{1}{-x} \cdot \frac{d}{dx}(-x)$$

$$= \frac{-1}{-x} = \boxed{\frac{1}{x}}$$

(b) $f(x) = \ln|\sec x + \tan x|$

$$f'(x) = \frac{1}{\sec x + \tan x} \cdot \frac{d}{dx}(\sec x + \tan x)$$

$$= \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}$$

$$\boxed{= \sec x}$$

It is often easier to first use the rules of logarithms to expand a logarithmic expression before taking the derivative. To do this properly you first must recognize when these rules can be applied and apply them correctly.

Rules and Non-Rules for Logarithms

- $\ln(AB) =$ $\ln(A) + \ln(B)$
- $\ln(A/B) =$ $\ln(A) - \ln(B$
- $\ln(A^r) =$ $r\ln(A)$
- $\ln(A + B) =$ non rule!
- $\ln(A - B) =$ non rule!
- $(\ln A)^r =$ non rule!

Example 5: Differentiate $g(x) = \log_5(x^2\sqrt{x+1})$ by first expanding the expressions using the rules for logarithms.

$$g(x) = 2\log_5 x + \frac{1}{2}\log_5(x+1)$$

$$g'(x) = \frac{2}{(\ln 5)x} + \frac{1}{2} \cdot \frac{1}{(\ln 5)(x+1)} = \boxed{\frac{2}{(\ln 5)x} + \frac{1}{2(\ln 5)(x+1)}}$$

Example 6: Differentiate $f(x) = \ln\left(\frac{x(x^2+1)^2}{\sqrt{2x^4-5}}\right)$

$$f(x) = \ln x + 2\ln(x^2+1) - \frac{1}{2}\ln(2x^4-5)$$

$$f'(x) = \frac{1}{x} + \frac{2}{x^2+1}\cdot 2x - \frac{1}{2}\frac{1}{2x^4-5}\cdot(8x^3)$$

$$\boxed{= \frac{1}{x} + \frac{4x}{x^2+1} - \frac{4x^3}{2x^4-5}}$$

Example 7: Differentiate the following functions.

(a) $f(x) = (\ln x)^5$ power rule first

$$f'(x) = 5(\ln x)^4 \cdot \frac{d}{dx}(\ln x)$$

$$= \boxed{\frac{5(\ln x)^4}{x}}$$

(b) $f(x) = \ln x^5$

MI

$$= 5\ln x$$

$$f'(x) = \boxed{\frac{5}{x}}$$

MII

$$f'(x) = \frac{1}{x^5}\cdot\frac{d}{dx}(x^5)$$

$$= \frac{1}{x^5}\cdot 5x^4 = \frac{5}{x}$$

Logarithmic Differentiation

Finding derivatives of complicated functions involving products, quotients and powers can often be simplified using logarithms. This technique is called logarithmic differentiation.

Example 8: Find the derivative of $y = \frac{x^7\sqrt{x^3+1}}{(5x+1)^4}$. ← agh!! Take ln at both sides.

$$\ln y = \ln\left(\frac{x^7\sqrt{x^3+1}}{(5x+1)^4}\right)$$

$$\ln y = 7\ln x + \frac{1}{2}\ln(x^3+1) - 4\ln(5x+1)$$

Impl. Diff!!

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{7}{x} + \frac{1}{2}\cdot\frac{1}{x^3+1}\cdot 3x^2 - \frac{4}{5x+1}\cdot 5$$

$$\frac{dy}{dx} = \left(\frac{7}{x} + \frac{3x^2}{2(x^3+1)} - \frac{20}{5x+1}\right)\cdot y$$

$$\boxed{\frac{dy}{dx} = \left(\frac{7}{x} + \frac{3x^2}{2(x^3+1)} - \frac{20}{5x+1}\right)\frac{x^7\sqrt{x^3+1}}{(5x+1)^4}}$$