Lecture: 3-8 Exponential Growth and Decay

In many natural phenomena, a quantity grows or decays at a rate proportional to their size. Suppose $y = f(t)$ is the number of individuals in a population at time t. Given an unlimited environment, adequate nutrition and immunity to disease it is reasonable to assume that the rate of growth is proportional to the population. That is,

constant of proportionality

population

$$f'(t) = \frac{dy}{dt} = ky$$

rate of growth

Example 1: Show that the equation $y = Ce^{kt}$ is a solution to the differential equation $\frac{dy}{dt} = ky$.

(a) Explain, in words, what it means for $y = Ce^{kt}$ to be a solution of the given differential equation.

when you "check" this in $dy/dt = ky$ it will "work".

(b) Show that $y = Ce^{kt}$ is a solution to the differential equation $\frac{dy}{dt} = ky$.

$$\frac{dy}{dt} = C \cdot e^{kt} \cdot k$$

$$\frac{dy}{dt} = k \cdot \underbrace{Ce^{kt}}$$

$$\frac{dy}{dt} = ky \; !$$

Theorem: The only solutions of the differential equation $dy/dt = ky$ are exponential functions of the form $y(t) = Ce^{kt}$ where $C = y(0)$

- Explain why $C = y(0)$.

$y(0) = Ce^{k(0)}$

$y(0) = C$;

This means that C is the initial population at time $t=0$.

- What does the constant k mean in this equation? What does the sign of k tell you about the growth of your population?

K is the rate of growth of the population;

If $K > 0$ the population is growing.

If $K < 0$ the population is decreasing.

Example 1: A bacteria culture $\boxed{\text{initially contains 10 cells}}$ and grows at a rate proportional to its size. After an hour the population has increased to 400.

(a) Find an expression for the number of bacteria after t hours. $f(0) = 10, \ f(1) = 400$

$$f(t) = Ce^{kt}$$

$$f(0) = Ce^{0}$$

$$10 = C$$

so $f(t) = 10e^{kt}$

$$f(1) = 400$$

$$400 = 10e^{k \cdot 1}$$

$$40 = e^{k}$$

$$k = \ln 40$$

Thus, $\boxed{f(t) = 10e^{(\ln 40)t} = 10 \cdot 40^{t}}$

(b) Find the number of bacteria after 3 hours.

$$f(3) = 10 \cdot 40^{3}$$

$$= \boxed{640{,}000 \text{ bacteria}}$$

(c) Find the rate of growth after 3 hours.

$$f'(x) = 10e^{\ln 40 t} \cdot \ln 40$$

$$f'(3) = 10 \cdot \ln 40 \cdot 40^{3}$$

$$= \boxed{2{,}360{,}882.8506 \text{ bacteria/hour}}$$

(d) When will the population reach 1,000?

$$1000 = 10e^{\ln 40 t}$$

$$100 = e^{\ln 40 t}$$

$$\ln 100 = \ln 40 t$$

$$t = \frac{\ln 100}{\ln 40}$$

$$\approx \boxed{1.248 \text{ hours}}$$

Example 2: Let $y = Ce^{kt}$ be the number of flies at time t, where t is measured in days. Suppose there are 100 flies after the second day and 400 flies after the fourth day. Assuming the growth rate is proportional to the population size find a model for this population's growth. When will the population of flies be 10,000?

$(2, 100)$ and $(4, 400)$

$100 = Ce^{2k} \Rightarrow C = \boxed{100/e^{2k}}$

$400 = Ce^{4k}$

$400 = \frac{100}{e^{2k}} e^{4k}$

$400 = 100e^{2k}$

$4 = e^{2k}$

$\ln 4 = 2k$

$\boxed{k = \frac{1}{2}\ln 4}$

$C = \frac{100}{e^{2\cdot\frac{1}{2}\ln 4}}$

$C = \frac{100}{4}$

$\boxed{C = 25}$

$y = 25e^{(\frac{1}{2}\ln 4)\cdot t}$

$= 25(e^{\ln 4^{1/2}})^t$

$\boxed{y = 25(2)^t}$

$100{,}000 = 25 \cdot 2^t$

$4000 = 2^t$

$\ln(4000) = \ln 2^t$

$\ln(4000) = t \ln 2$

$t = \frac{\ln 4000}{\ln 2} \approx \boxed{11.966 \text{ days}}$

Example 3: The half-life of cesium-137 is 30 years. The 1986 explosion at Chernobyl sent about 1000 kg of radioactive cesium-137 into the atmosphere.

(a) Find the mass that remains after t years.

$\boxed{m(t) = 1000\left(\tfrac{1}{2}\right)^{(t/30)}}$

$m(t) = Ce^{kt}$

$m(t) = 1000e^{kt}$

Know $t = 30$, $m(30) = 500$

$500 = 1000e^{30k}$

$\frac{1}{2} = e^{30k}$

$\ln(\frac{1}{2}) = 30k$

$k = \frac{1}{30}\ln(\frac{1}{2})$

$m(t) = 1000e^{\frac{1}{30}\ln(1/2)t}$

$= 1000\left(e^{\ln(1/2)^{1/30}}\right)^t$

$= 1000\left(\left(\tfrac{1}{2}\right)^{1/30}\right)^t$

$= 1000\left(\tfrac{1}{2}\right)^{t/30}$

(b) If even 100 kg remains in Chernobyl's atmosphere, the area is considered unsafe for human habitation. Determine when Chernobyl will be safe.

$100 = 1000\left(\tfrac{1}{2}\right)^{t/30}$

$10 = \left(\tfrac{1}{2}\right)^{t/30}$

$\ln(\frac{1}{10}) = \ln\left(\tfrac{1}{2}\right)^{t/30}$

$\ln(\frac{1}{10}) = \frac{t}{30}\ln(\frac{1}{2})$

$t = 30\ln(\frac{1}{10}) / \ln(\frac{1}{2}) \approx 99.658 \approx$

100 years after 1986, so $\boxed{2086}$

Example 4: A sample of radioactive tritium-3 decayed to 95% of its original amount after a year.

(a) What is the half-life of tritium-3?

$y = C\left(\frac{1}{2}\right)^{(t/h)}$

$t=1,\ y=(0.95)C$

so, $0.95C = C(0.5)^{1/h}$

$0.95 = (0.5)^{1/h}$

$\ln(0.95) = \ln(0.5)^{1/h}$

$\ln(0.95) = \frac{1}{h}\ln(0.5)$

$h\ln(0.95) = \ln(0.5)$

$h = \ln 0.5 / \ln 0.95$

$\boxed{h \approx 13.513}$

(b) How long would it take the sample to decay to 10% of its original amount?

$0.1C = C(0.5)^{t/13.513}$

$0.1 = (0.5)^{t/13.513}$

$\ln(0.1) = \frac{t}{13.513}\ln(0.5)$

$t = \frac{13.513\ln(0.1)}{\ln(0.5)}$

$\boxed{t \approx 44.891 \text{ years}}$

Example 5: Scientists can determine the age of ancient objects by the method of *radiocarbon dating*. The bombardment of the upper atmosphere by cosmic rays converts nitrogen to a radioactive isotope of carbon, ^{14}C, with a half life of about 5730 years. Vegetation absorbs carbon dioxide through the atmosphere and animal life assimilates ^{14}C through food chains. When a plant or animal dies, it stops replacing its carbon and the amount of ^{14}C begins to decrease through radioactive decay. Therefore, the level of radioactivity must also decay exponentially.
A parchment fragment was discovered that had about 74% as much ^{14}C radioactivity as does the plant material on earth today. Estimate the age of the parchment.

$f(t) = C\left(\frac{1}{2}\right)^{t/5730}$

Q: when is $f(t) = 0.74C$?

$0.74C = C(0.5)^{t/5730}$

$0.74 = (0.5)^{t/5730}$

$\ln(0.74) = \ln(0.5)^{t/5730}$

$\ln(0.74) = \frac{t}{5730}\ln(0.5)$

$t = \frac{5730\ln(0.74)}{\ln(0.5)}$

$\boxed{t \approx 2489.13 \text{ years old @ discovery}}$

Newton's Law of Cooling

The rate of cooling (or warming) of an object is proportional the temperature difference between the object and its surroundings: $\frac{dT}{dt} = k(T - T_s)$

temp → T ; temp of surroundings → T_s

Example 6: When a cold drink is taken from a refrigerator, its temperature is 40° F. After 25 minutes in a 70°F room its temperature has increased to 52°F.

(a) What is the temperature of the drink after 50 minutes?

not quite $\frac{dy}{dt} = ky$; close → let $y = T - T_s$, this works

Have $y = T - 70$ here ; $y(0) = 40 - 70 = -30$

Thus $y(t) = -30e^{kt}$, also after $t = 25$ min drink is 52°F

so $y(52) = 52 - 70 = -18$ and . . .

$$-18 = -30e^{25k}$$
$$9/15 = e^{25k}$$
$$25k = \ln(9/15)$$
$$k = \ln(9/15)/25$$

$$y(t) = -30e^{(\frac{1}{25}\ln(9/15))t}$$
$$T - 70 = -30e^{1/25 \ln(9/15 t)}$$
$$\boxed{T(t) = 70 - 30e^{1/25 \ln(9/15)t}}$$
$$\boxed{T(50) = 59.2°F}$$

(b) When will its temperature reach 65°F?

$$65 = 70 - 30e^{1/25 \ln(9/15)t}$$
$$-5 = -30e^{1/25 \ln(9/15)t}$$
$$1/6 = e^{1/25 \ln(9/15)t}$$
$$\ln(1/6) = 1/25 \ln(9/15 t)$$
$$\frac{25\ln(1/6)}{\ln(9/15)} = t$$

$$\boxed{t \simeq 87.689 \text{ min}}$$

(c) What happens to the temperature of the drink as $t \to \infty$? Is this expected?

$$\lim_{t\to\infty} T(t) = \lim_{t\to\infty}(70 - 30e^{-0.6385t})$$
$$= \lim_{t\to\infty}(70 - 30/e^{0.6385t})$$
$$= \boxed{70°F}$$

← as $t \to \infty$, $T(t) \to 70°F$ or the ambient room temp