LECTURE: 3-9 RELATED RATES

Example 1:

- The cost of producing x ounces of gold from a new gold mine is $C = f(x)$ dollars. What is the meaning of the derivative $f'(x)$?

$f'(x)$ has units \$/ounce $\rightarrow$ this describes how costs change relative to production level.

- The number of bacteria after t hours in a controlled laboratory is $n = f(t)$. What is the meaning of $f'(x)$?

$f'(t)$ has units # of bacteria/hour. This describes how the bacteria population is changing at any specific moment in time

So far, we've implicitly differentiated a function with respect to a variable that is somehow part of the equation. Today, we're going to implicitly differentiate with respect to a third variable which usually is time.

Example 2: If $y = x^4 + 2x^3$ and $dx/dt = 7$, find dy/dt when $x = 1$. (Now, you have to first differentiate **then** use the information that you have.)

$$\frac{d}{dt}(y) = \frac{d}{dt}(x^4 + 2x^3)$$

$$\frac{dy}{dt} = 4x^3\frac{dx}{dt} + 6x^2\frac{dx}{dt}$$

know $dx/dt = 7$, want dy/dt at $x=1$

$$\frac{dy}{dt} = 4(1)^3(7) + 6(1)^2(7)$$

$$= 28 + 42$$

$$= \boxed{70}$$

Example 3: A pebble dropped into a calm pond, causing ripples in the form of circles. The radius r of the outer ripple is increasing at a constant rate of 1 foot per second. When the radius is 4 feet, at what rate is the area A of the water disturbed changing?

know $\frac{dr}{dt} = 1$ ft/sec

want to know $\frac{dA}{dt}$ when $r=4$.

① relate A + r in an equation:

$$A = \pi r^2$$

② Take derivative w/respect to t

$$\frac{d}{dt}A = \frac{d}{dt}(\pi r^2)$$

$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$$

③ Input info + solve

$$\frac{dA}{dt} = 2\pi(4)(1)$$

$$= \boxed{8\pi \text{ ft}^2/\text{sec}}$$

Example 4: Air is being pumped into a spherical balloon so that it's volume increases at a rate of 4.5 ft^3/min. How fast is the radius of the balloon increasing when the diameter is 4 ft?

① know $\frac{dV}{dt} = 4.5\ \text{ft}^3/\text{min}$; want dr/dt when $r = 2$

② equation relating V and r is: $V = \frac{4}{3}\pi r^3$

③ derivative w/respect to t: $\frac{d}{dt} V = \frac{d}{dt} \frac{4}{3}\pi r^3$

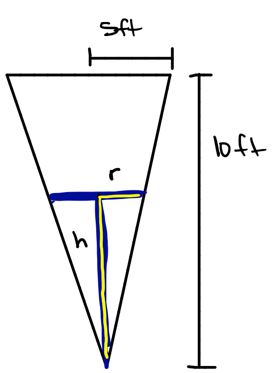
$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

④ Input: $4.5 = 4\pi(2^2)\frac{dr}{dt}$

$$\frac{9}{2} = 16\pi \frac{dr}{dt}$$

$$\boxed{\frac{dr}{dt} = \frac{9}{32\pi}\ \text{ft}/\text{min}}$$

Example 5: Water runs into a conical tank at the rate of 9 ft^3/min. The tank stands point down and has a height of 10 ft and a base radius of 5 ft. How fast is the water level rising when the water is 6 ft deep?



① know $dV/dt = 9\ \text{ft}^3/\text{min}$, want dh/dt when $h = 6$

② Formula: $V = \frac{\pi}{3} r^2 h$ ← we want only h's use geometry!

know: $\frac{r}{h} = \frac{5}{10} \Rightarrow r = \frac{1}{2}h$

Thus $V = \frac{\pi}{3}\left(\frac{1}{2}h\right)^2 h$

$$V = \frac{\pi}{12} h^3$$

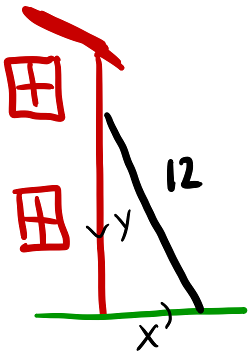
③ Take deriv: $\frac{dV}{dt} = \frac{3\pi}{12} h^2 \frac{dh}{dt}$

$$\frac{dV}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt}$$

④ Input $9 = \frac{\pi}{4} \cdot 6^2 \frac{dh}{dt}$

$$9 = 9\pi \frac{dh}{dt} \Rightarrow \boxed{\frac{dh}{dt} = \frac{1}{\pi}\ \text{ft}/\text{min}}$$

Example 6: A ladder 12 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?



① Know $\frac{dx}{dt} = 1$ ft/sec, want $\frac{dy}{dt}$ when $x=6$

② $x^2 + y^2 = 12^2 \Rightarrow x^2+y^2 = 144$

③ $2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \div$ by $2 \Rightarrow x\frac{dx}{dt} + y\frac{dy}{dt} = 0$

④ $6(1) + y\,(dy/dt) = 0$

$6 + 6\sqrt{3}\ dy/dt = 0$

$dy/dt = -6/6\sqrt{3}$

$= -1/\sqrt{3}$

$= \boxed{-\sqrt{3}/3 \text{ ft/sec}}$

must find y!

$x^2+y^2 = 144$ when $x=6$

$6^2 + y^2 = 144$

$36 + y^2 = 144$

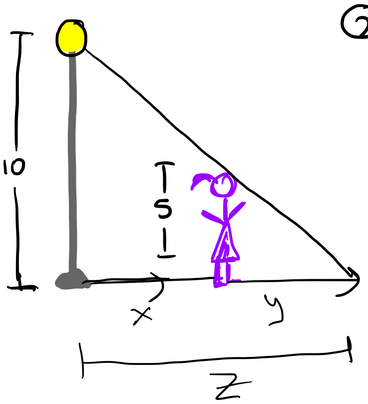
$y^2 = 108$

$y = \sqrt{108} = \sqrt{9 \cdot 4 \cdot 3}$

$= \sqrt{9(12)} = 6\sqrt{3}$

Example 7: A street light is mounted at the top of a 10-ft-tall pole. A woman 5 ft tall walks away from the pole along a straight path at a speed of 5 ft/s. How fast is the tip of her shadow moving when she is 40 ft from the pole?

① know $dx/dt = 5$, want $\frac{dz}{dt}$ when $x = 40$



② $x + y = z$ ← need to get rid of y

use similar triangles

$\frac{10}{x+y} = \frac{5}{y} \Rightarrow 10y = 5x + 5y$

$\Rightarrow 5y = 5x$

$\Rightarrow y = x$

so → $x + x = z$

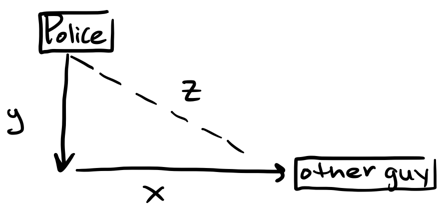
$\Rightarrow 2x = z$

③ $2\frac{dx}{dt} = \frac{dz}{dt}$

④ $2(5) = dz/dt$

$\boxed{\frac{dz}{dt} = 10 \text{ ft/sec}}$

Example 8: A police cruiser, approaching a right-angled intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east. When the cruiser is 0.6 mi north of the intersection and the car is 0.8 mi to the east, the police determine that the distance between them and the car they are chasing is increasing at a rate of 20 mph. If the cruiser is moving at 60 mph at the instant of measurement, what is the speed of the car?



① know $dy/dt = -60$, $dz/dt = 20$

want dx/dt when $y = 0.6$, $x = 0.8$, $z = \sqrt{0.6^2 + 0.8^2} = \sqrt{0.36 + 0.64} = \sqrt{1} = 1$

② $x^2 + y^2 = z^2$

③ $2x \frac{dx}{dt} + 2y\, dy/dt = 2z\, dz/dt$

$x\, dx/dt + y\, dy/dt = z\, dz/dt$

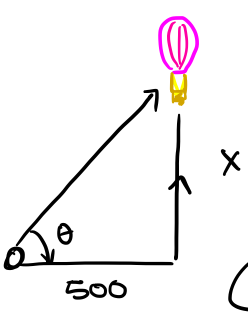
$(0.8)\, dx/dt + 0.6(-60) = 1(20)$

$8/10\ dx/dt - 36 = 20$

$dx/dt = 56 \cdot 10/8 = \boxed{70 \text{ mph}}$

Example 9: A hot air balloon rising straight up from a level field is tracked by a range finder 500 feet from the lift-off point. At the moment the range finder's elevation angle is $\pi/4$, the angle is increasing at the rate of 0.14 radians/min. How fast is the balloon rising at that moment?

① know $d\theta/dt = 0.14$ rad/min. want dx/dt when $\theta = \pi/4$.



② $\tan\theta = \frac{x}{500} \Rightarrow 500 \tan\theta = x$

③ $500 \sec^2\theta \frac{d\theta}{dt} = 1 \frac{dx}{dt}$

④ $\frac{dx}{dt} = 500 \left(\frac{1}{\cos \pi/4}\right)^2 \cdot \frac{14}{100}$

$= 500 \left(1/(\sqrt{2}/2)\right)^2 \cdot \frac{14}{100}$

$= 5(14)\left(1/(1/2)\right)$

$= 5(14)(2)$

$= \boxed{140 \text{ ft/min}}$