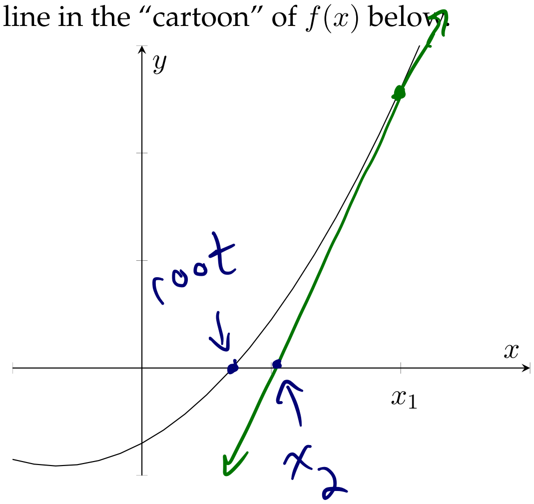
LECTURE NOTES: 4-8 NEWTON'S METHOD

MOTIVATING QUESTION: Suppose we wanted to find the x-intercepts of $f(x) = x - 2\sin x$. From the graph (or the ~~Mean Value Theorem~~) we can see there exists a positive (and negative) solution. How to find it?

IVT!

DERIVATION OF NEWTON'S METHOD:

1. Write the equation of the line tangent to the curve $y = f(x)$ at the x-value x_1. Sketch the tangent line in the "cartoon" of $f(x)$ below.



$m = f'(x_1)$ slope

$(x_1, f(x_1))$ point

Line

$$y - f(x_1) = f'(x_1)(x - x_1)$$

$$y = f'(x_1)(x - x_1) + f(x_1)$$

2. In your picture above, label the x-value where the tangent line intersects the x-axis as x_2.

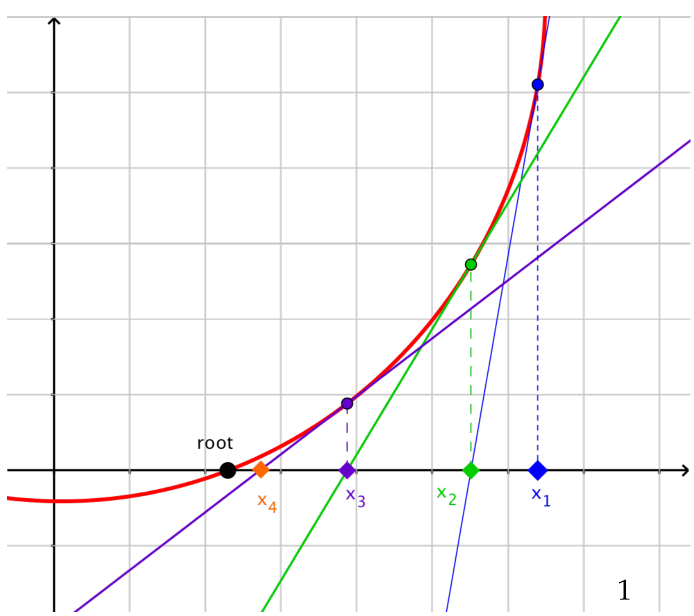
3. Solve for x_2 using your equation from part (1) above.

plug in $(x_2, 0)$ (why $y=0$?)

$$0 = f'(x_1)(x_2 - x_1) + f(x_1)$$

solving for x_2: $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

GEOMETRIC EXPLANATION OF NEWTON'S METHOD:



nothing special about 1, 2!

Formula for Newton's Method:

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

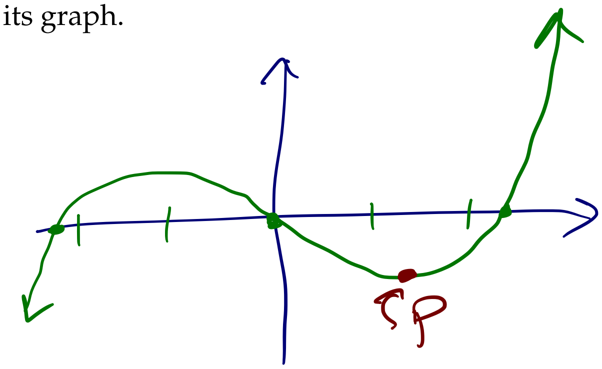
Model Problem: Let $f(x) = x^3 - 5x$.

1. Factor $f(x)$, find its roots algebraically, and sketch its graph.

$f(x) = x(x^2-5)$

$= x(x+\sqrt{5})(x-\sqrt{5})$

$x = 0, -\sqrt{5}, \sqrt{5}$



2. Assume you couldn't factor the function and you wanted to find its positive root. What would be a reasonable first guess and why?

$x=3 : f(3) = 12 > 0$

$x=2 : f(2) = -2 < 0$

So any x value between 2 & 3 would be a good starting point!

3. Using a first guess of $x_1 = 3$, calculate 2 iterations of Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_2 = 3 - \frac{f(3)}{f'(3)} = 3 - \frac{12}{22} = 2.\overline{45}$$

$$x_3 = 2.\overline{45} - \frac{f(2.\overline{45})}{f'(2.\overline{45})} \approx 2.262154$$

4. How close is your estimate of the root, x_3, to the actual root?

So $\sqrt{5} \approx 2.236068$ difference: $\boxed{0.0260}$!

5. How important is the first guess (part 2. above)? In particular, are there any truly *bad* guesses that won't get to our sought after root?

Look at point P above with coor. $(x_p, f(x_p))$

$f'(x_p) = 0$! Tangent line doesn't intersect x-axis.

EXAMPLE 1: Approximate any zero of $f(x) = x - 2\sin x$ using 2 iterations of Newton's Method. Graph $f(x)$ and draw the first iteration.

Clearly $x=0$ is a zero!

IVT

$g(\pi/2) = \pi/2 - 2 < 0$

$g(\pi) = \pi > 0$

As $\pi/2 \approx 1.5$ and $\pi \approx 3$, there is a root between 1.5 and 3.

Start with $x_1 = 2$.

$x_1 = 2 \qquad f'(x) = 1 - 2\cos x$

$$x_2 = 2 - \frac{f(2)}{f'(2)}$$

$$= 2 - \frac{(2 - 2\sin 2)}{1 - 2\cos 2}$$

$$\approx 1.900996$$

$$x_3 \approx (1.9\ldots) - \frac{f(1.9\ldots)}{f'(1.9\ldots)}$$

$$\boxed{\approx 1.8955}$$

EXAMPLE 2: Estimate $\sqrt[6]{7}$ correct to 5 decimal places.

Hmm... need a function!

Observe $f(x) = x^6 - 7$ has root $\sqrt[6]{7}$!

a little bigger than 1?

$f'(x) = 6x^5$

$x_1 = 1.1 \leftarrow$ choose

$$x_2 = 1.1 - \frac{f(1.1)}{f'(1.1)} \approx 1.64174877$$

$x_3 = \ldots \approx 1.46558165$

$x_4 \approx 1.39386642$

$x_5 \approx 1.383293572$

$x_6 \approx 1.\underline{38308}7631$

$x_7 = 1.\underline{383087}5$

$$\boxed{\sqrt[6]{7} \approx 1.383087}$$