LECTURE NOTES: 4-9 ANTIDERIVATIVES

MOTIVATING IDEAS:

1. If $s(t)$ give the position of an object at each time t, then $s'(t)$ is velocity and $s''(t)$ is acceleration.

2. If $P(t)$ is the number of individuals in a population at each time t, then $P'(t)$ is population growth rate.

- The point: Until now, we start w/ position and find velocity.
  What if we know the velocity of an object, can we determine position?
- More abstractly, can we "undo" differentiation?

Definition: Given a function $f(x)$, any function F such that $F'(x) = f(x)$ is called an antiderivative of $f(x)$.

EXAMPLE: Find three different antiderivatives of $f(x) = x^2$.

$F(x) = \frac{1}{3}x^3$, $F(x) = \frac{1}{3}x^3 + 1$, $F(x) = \frac{1}{3}x^3 - 21.37$

QUESTION 1: How to any two antiderivatives of $f(x) = x^2$ differ?

They have different constant terms.

QUESTION 2: How can you characterize *all* antiderivatives of $f(x) = x^2$ simultanously? Explain what your expression means.

$F(x) = \frac{1}{3}x^3 + C$, where C could be any real number.

QUESTION 3: Fill in the blank:

Theorem: If F is an anti-derivative of f on an interval I, then the most general anti-derivative of f on I is $F(x) + C$.

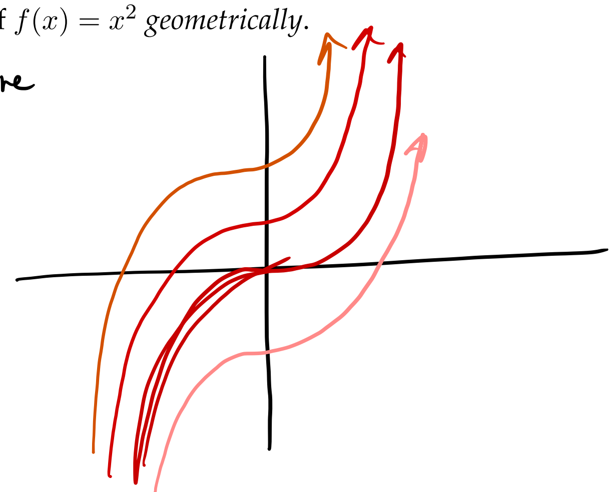
QUESTION 4: Do you really believe this Theorem? Here's a check. It's easy to see that $F(x) = x^2 - \pi$ is an antiderivative of $f(x) = 2x$. (Right?) Apply the Theorem in Question 3 to this choice of F.

The theorem says the most general expression for antiderivatives of $f(x)$ is: $\boxed{x^2 - \pi + C}$

QUESTION 5: Describe the family of antiderivatives of $f(x) = x^2$ *geometrically*.

The collection of all anti derivatives of $f(x) = x^2$ consist of vertical translations of the graph $y = \frac{1}{3}x^3$.

picture



PRACTICE PROBLEMS:

1. Minnie Mouse says that $F(x) = 5x^{2/3} + x + \sqrt{2}$ is an antiderivative of $f(x) = \frac{10+3\sqrt[3]{x}}{3\sqrt[3]{x}}$. Find the *most efficient way* to determine if she is correct.

Take the derivative + simplify.

$F'(x) = \frac{10}{3}x^{-1/3} + 1 = \frac{10}{3\sqrt[3]{x}} + 1$

$= \frac{10}{3\sqrt[3]{x}} + \frac{3\sqrt[3]{x}}{3\sqrt[3]{x}} = \frac{10 + 3\sqrt[3]{x}}{3\sqrt[3]{x}}$.

Minnie Mouse is right.

2. Fill in the table below. Assume n and a are fixed constants.

Function	Particular Anti-derivative	Function	Particular Anti-derivative
x^n $(n \neq -1)$	$x^{n+1}/n+1$	$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x$
$\cos x$	$\sin x$	$\frac{1}{1+x^2}$	$\arctan x$
$\sin x$	$-\cos x$	$\frac{1}{x}$	$\ln\lvert x\rvert$
$\sec^2 x$	$\tan x$	e^x	e^x
$\csc^2 x$	$-\cot x$	a^x	$a^x/\ln a$
$\sec x \tan x$	$\sec x$	$\csc x \cot x$	$-\csc x$

3. Assuming $F(x)$ is an antiderivative of $f(x)$ and $G(x)$ is an antiderivative of $g(x)$, fill in the blanks below.

   (a) For any constant a the general antiderivative of $af(x)$ is $aF(x) + C$.

   (b) The general antiderivative of $f(x) + g(x)$ is $F(x) + G(x) + C$.

4. Find the most general antiderivative of each function below and then check your answer.

$-t^{-2}$

(a) $f(x) = x^{20} + 4x^{10} + 8$

$F(x) = \frac{1}{21}x^{21} + \frac{4}{11}x^{11} + 8x + C$

Check

$F' = x^{20} + 4x^{10} + 8$ ✓

(b) $f(t) = \frac{5 \sec t \tan t}{3} - 4 \sin t - \frac{1}{t^2} + e^2$

$F(t) = \frac{5}{3}\sec t + 4\cos t + t^{-1} + e^2 t + C$

Check:

$F'(t) = \frac{5}{3}\sec t \tan t - 4\sin t - 1 \cdot t^{-2} + e^2$ ✓

(c) $g(x) = x(2x^5 + \sqrt{x}) + \sqrt{2}x$

$g(x) = 2x^6 + x^{3/2} + \sqrt{2}\,x$

$G(x) = \frac{2}{7}x^7 + \frac{2}{5}x^{5/2} + \frac{\sqrt{2}}{2}x^2 + C$

(d) $f(t) = \frac{3x^7 - \sqrt{x}}{x^2} = 3x^5 - x^{-3/2}$

$F(t) = \frac{3}{6}x^6 - (-2)x^{-1/2} + C$

$= \frac{1}{2}x^6 + 2x^{-1/2} + C$

(e) $g(x) = 8\left(\frac{e^x}{5} - \frac{5}{x^2+1}\right)$

$G(x) = 8\left(\frac{1}{5}e^x - 5\arctan x\right) + C$

(f) $s(t) = \frac{9t^3 - 4t^{5/3} - 4}{t} = 9t^2 - 4t^{2/3} - 4t^{-1}$

$S(t) = \frac{9}{3}t^3 - 4\left(\frac{3}{5}\right)t^{5/3} - 4\ln|t| + C$

5. Given $f'(x) = x\sqrt{x}$ and $f(1) = 2$, find $f(x)$. **Note:** The directions are different here. You are not asked to find a *family* of antiderivatives but a *particular* antiderivative.

$f'(x) = x^{3/2}$

$f(x) = \frac{2}{5}x^{5/2} + C$

$2 = f(1) = \frac{2}{5}(1)^{5/2} + C$

So $C = 2 - \frac{2}{5} = \frac{8}{5}$

Answer

$f(x) = \frac{2}{5}x^{5/2} + \frac{8}{5}$

6. Explain *geometrically* what piece of information you are given in the previous problem that allows you to identify a particular member of the family of antiderivatives.

We are given a point on the graph of $f(x)$. So, among all the vertical translations, we can pick out a single one.

7. Find (the particular function) $f(x)$ assuming:

- $f''(x) = \sqrt[3]{x} = x^{1/3}$
- $f'(8) = 1$ and $f(1) = -6$.

So $f'(x) = \frac{3}{4}x^{4/3} + C$

$1 = f'(8) = \frac{3}{4}(8)^{4/3} + C$

So $C = 1 - 12 = -11$

So $f'(x) = \frac{3}{4}x^{4/3} - 11$.

Now $f(x) = \frac{3}{4} \cdot \frac{3}{7} x^{7/3} - 11x + C$

$f(x) = \frac{9}{28}x^{7/3} - 11x + C$

$-6 = f(1) = \frac{9}{28} - 11 + C$

So $C = 5 - \frac{9}{28} = \frac{140-9}{28} = \frac{131}{28}$

ANS:

$f(x) = \frac{9}{28}x^{7/3} - 11x + \frac{131}{28}$

8. A particle moves in a straight line and has acceleration given by $a(t) = 5\cos t - 2\sin t$. Its initial velocity is $v(0) = -6$ m/s and its initial position is $s(0) = 2$ m. Find its position function $s(t)$.

$v(t) = 5\sin t + 2\cos t + C$

$-6 = v(0) = 0 + 2 + C$

So $C = -8$.

Now $v(t) = 5\sin t + 2\cos t - 8$

$s(t) = -5\cos t + 2\sin t - 8t + C$

$2 = s(0) = -5 + 0 + 0 + C$

$C = 7$

answer:

$s(t) = -5\cos t + 2\sin t - 8t + 7$

9. A ball is thrown upward with a speed of 48 ft/s from the edge of a cliff 432 feet above the ground. Find its height above the ground t seconds later. When does it reach its maximum height? When does it hit the ground?

hint: acceleration due to gravity is $32\ ft/s^2$. So $a(t) = -32\ ft/sec^2$

$\left.\begin{array}{l} v(0) = 48\ ft/s \\ s(0) = 432\ ft \end{array}\right\}$ initial conditions

If $v'(t) = a(t) = -32$, then $v(t) = -32t + C$.

using •, $48 = -3t(0) + C$. So $48 = C$. So $\boxed{v(t) = -32t + 48}$.

If $s'(t) = v(t) = -32t + 48$, then $s(t) = -16t^2 + 48t + C$.

Using •, $432 = s(0) = -16\cdot 0^2 + 48\cdot 0 + C$. So $432 = C$. So $s(t) = -16t^2 + 48t + 432$.

• max height when? At t (time) when $V = 0$.

So $0 = -32t + 48$, or, $t = 48/32 = \frac{3}{2}$. Ans: The maximum height occurs after 1.5 seconds.

• hit ground when? At t when $s = 0$.

$0 = -16t^2 + 48t + 432 = -16(t^2 - 3t - 27)$. or $\boxed{t = 6.9 \text{ seconds}}$