Lecture Notes: Review of Chapters 3 & 4

1. A few warm-up derivatives. Simplify your answers.

(a) Given $f(x) = \arcsin(e^{3x})$, find $f'(x)$.

$$f'(x) = \frac{1}{\sqrt{1-(e^{3x})^2}} \cdot e^{3x} \cdot 3$$

$$= \frac{3e^{3x}}{\sqrt{1-e^{6x}}}$$

(b) Find dy/dx if $\cos(xy) = x^2 - y$. Implicit Diff.

$$[-\sin(xy)] \cdot (y + xy') = 2x - y'$$

$$-y\sin(xy) - (x\sin(xy)) \cdot y' = 2x - y'$$

$$-y\sin(xy) - 2x = [x\sin(xy) - 1]\, y'$$

So $$\frac{dy}{dx} = -\frac{y\sin(xy) + 2x}{x\sin(xy) - 1}$$

(c) Find y' if $y = \frac{(2x+1)^3}{\sqrt{4+x^2}}$. logarithmic diff. too.

So $\ln y = 3\ln(2x+1) - \frac{1}{2}\ln(4+x^2)$.

So $\frac{1}{y} \cdot y' = \frac{6}{2x+1} - \frac{1}{2}\frac{2x}{4+x^2}$.

So $y' = y\left(\frac{6}{2x+1} - \frac{x}{4+x^2}\right)$

ANS:

$$y' = \left(\frac{(2x+1)^3}{\sqrt{4+x^2}}\right)\left(\frac{6}{2x+1} - \frac{x}{4+x^2}\right)$$

(d) Find $g'(x)$ if $g(x) = (\cos x)^x$.

$\ln y = x\ln(\cos x)$. So $\frac{1}{y} \cdot y' = 1 \cdot \ln(\cos x) + x \cdot \frac{(-\sin x)}{\cos x}$

So $y' = y\left(\ln(\cos x) - \frac{x\sin x}{\cos x}\right)$

ANS:

$$g'(x) = (\cos x)^x(\ln(\cos x) - x\tan x)$$

2. Find the equation of the line tangent to $f(x) = \frac{1+\cos x}{1-\cos x}$ when $x = \pi/2$.

Point: when $x = \frac{\pi}{2}$, $y = \frac{1+\cos \pi/2}{1-\cos \pi/2} = \frac{1}{1} = 1$

$(\frac{\pi}{2}, 1)$

Slope: $$f'(x) = \frac{(1-\cos x)(-\sin x) - (1+\cos x)(\sin x)}{(1-\cos x)^2}$$

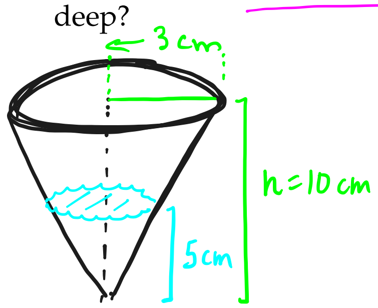
$$= \frac{-2\sin x}{(1-\cos x)^2}.$$

When $x = \frac{\pi}{2}$, $m = f'(\pi/2) = \frac{-2\sin(\pi/2)}{(1-\cos(\pi/2))^2} = -2$

So $y - 1 = -2(x - \frac{\pi}{2})$

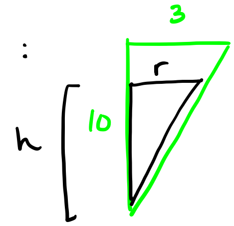
ANS: $y = -2x + (\pi + 1)$

3. A paper cup has the shape of a cone with height 10 cm and radius 3 cm (at the top). If water is poured into the cup at a rate of $2\,\text{cm}^3/\text{sec}$, how fast is the water level rising when the water is 5 cm deep?



Volume of cone: $V = \frac{1}{3} \cdot \pi r^2 \cdot h$

Use similar triangles:



So $\frac{r}{h} = \frac{3}{10}$ or $r = \frac{3}{10} h$

So $V = \frac{\pi}{3} \left(\frac{3}{10} h\right)^2 \cdot h = \frac{3\pi}{100} h^3$.

Now take derivative implicitly with respect to time t in seconds:

$$\frac{dV}{dt} = \frac{9\pi}{100} \cdot h^2 \cdot \frac{dh}{dt}.$$

Know: $\frac{dV}{dt} = 2$, $h = 5$; want: $\frac{dh}{dt}$.

plug-n-chug:

$$2 = \frac{9\pi}{100} \cdot 25 \cdot \frac{dh}{dt} \quad \text{or} \quad \frac{dh}{dt} = \frac{8}{9\pi}$$

ANS: The water is rising at a rate of $\frac{8}{9\pi}$ cm/sec when the water is 5 cm deep.

4. Find the linearization of $f(x) = \sqrt{x}$ at $a = 4$ and use it to estimate $\sqrt{4.1}$.

translation: use tangent line of $f(x)$ at $x=4$ to estimate $f(4.1)$

① Find tangent line

point: $(4,2)$

slope: $f(x) = x^{1/2}$, $f'(x) = \frac{1}{2} x^{-1/2}$

$$m = f'(4) = \frac{1}{2} \cdot \frac{1}{\sqrt{4}} = \frac{1}{4} = 0.25$$

line: $y - 2 = \frac{1}{4}(x-4)$ or

$$L(x) = \underbrace{\frac{1}{4}(x-4) + 2}_{} = \frac{1}{4}x + 1$$

I will use this version.

② Estimation:

$$L(4.1) = \frac{1}{4}(4.1-4) + 2 = (0.25)(0.1) + 2$$

$$= 2.025$$

Note: ① didn't need a calculator at all!

② $\sqrt{4.1} \approx 2.02484567$:)

5. (a) What are critical numbers of a function f?

x-values in the domain of f so that

$f'(x)=0$ or $f'(x)$ is undefined.

(b) How do you find the absolute maximum and minimum of a continuous function f on a closed interval?

① Check the y-values of endpoints & critical points

② Largest y-value = max, smallest y-value = min.

(c) Find the critical numbers of $f(x) = \sin x + \cos^2 x$ in $[0, \pi]$.

$f'(x) = \cos x + 2\cos x(-\sin x)$ ← f' is always defined!

$= \cos x(1-2\sin x) = 0$

$\cos x = 0$ when $x = \pi/2$

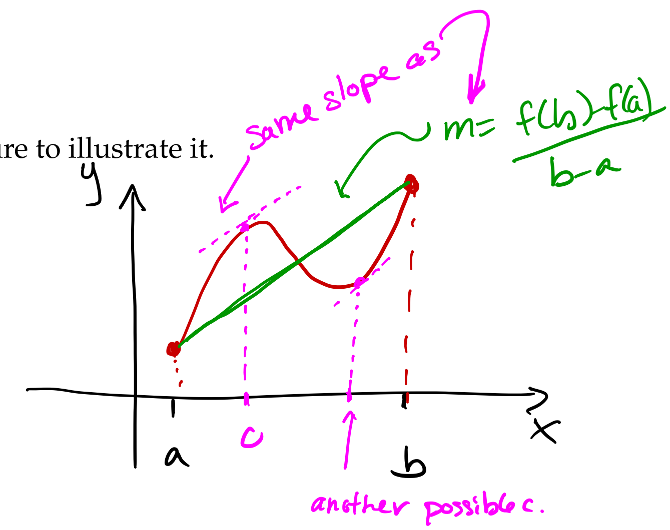
$1 - 2\sin x = 0$ when $\sin x = \frac{1}{2}$. That is, for $x = \frac{\pi}{6}, \frac{5\pi}{6}$.

ANS: The critical numbers of $f(x)$ are $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$

6. (a) State the Mean Value Theorem and draw a picture to illustrate it.

If $f(x)$ is continuous + differentiable on $[a,b]$, then there is some x-value c in (a,b) so that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$



(b) Determine whether the Mean Value Theorem applies to $f(x) = x(x^2 - x - 2)$ on $[-1, 1]$. If it can be applied find all numbers that satisfy the conclusion of the Mean Value Theorem.

$f(x) = x^3 - x^2 - 2x$, $f'(x) = 3x^2 - 2x - 2$

$f(-1) = (-1)^3 - (-1)^2 - 2(-1) = -1 - 1 + 2 = 0$

$f(1) = 1 - 1 - 2 = -2$

$$\frac{f(b)-f(a)}{b-a} = \frac{f(1)-f(-1)}{1-(-1)} = \frac{-2-0}{2} = -1$$

We need to find c so that $f'(c) = -1$.

or $3c^2 - 2c - 2 = -1$

So, $3c^2 - 2c - 1 = 0$ or

$(3c+1)(c-1) = 0$. Thus, $c = -\frac{1}{3}, 1$.

Only $c = -1/3$ is in $(-1, 1)$.

Answer: $c = -1/2$ is the only number satisfying the MVT for $f(x)$ on $[-1, 1]$.

7. Let $f(x) = 2x - 2\cos x$ on $[-\pi, 2\pi]$

(a) Find the open intervals on which the function is increasing or decreasing.

$f'(x) = 2 + 2\sin x = 0$

$\sin x = -1$

So $x = -\frac{\pi}{2}, \frac{3\pi}{2}$

sign of f'

$++\ 0\ +\ +\ ++\ 0\ ++$

$-\pi \quad -\frac{\pi}{2} \quad 0 \quad \pi \quad \frac{3\pi}{2} \quad 2\pi$

$\boxed{f \text{ is always increasing}}$

Note: horizontal tangents here.

(b) Apply the first derivative test to identify all relative extrema. Classify each as a local maxima or local minima.

Since f is always increasing on $[-\pi, 2\pi]$, f has a (local+absolute) min of $f(-\pi) = -2\pi + 2$ and a (local+absolute) max of $f(2\pi) = 4\pi + 2$.

(c) Find the open intervals on which the function is concave up or concave down.

$f''(x) = 2\cos x = 0$

when $x = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$

$-\ 0\ +\ 0\ -\ 0\ +$ ← sign of f''

$-\pi \quad -\frac{\pi}{2} \quad 0 \quad \frac{\pi}{2} \quad \pi \quad \frac{3\pi}{2} \quad 2\pi$

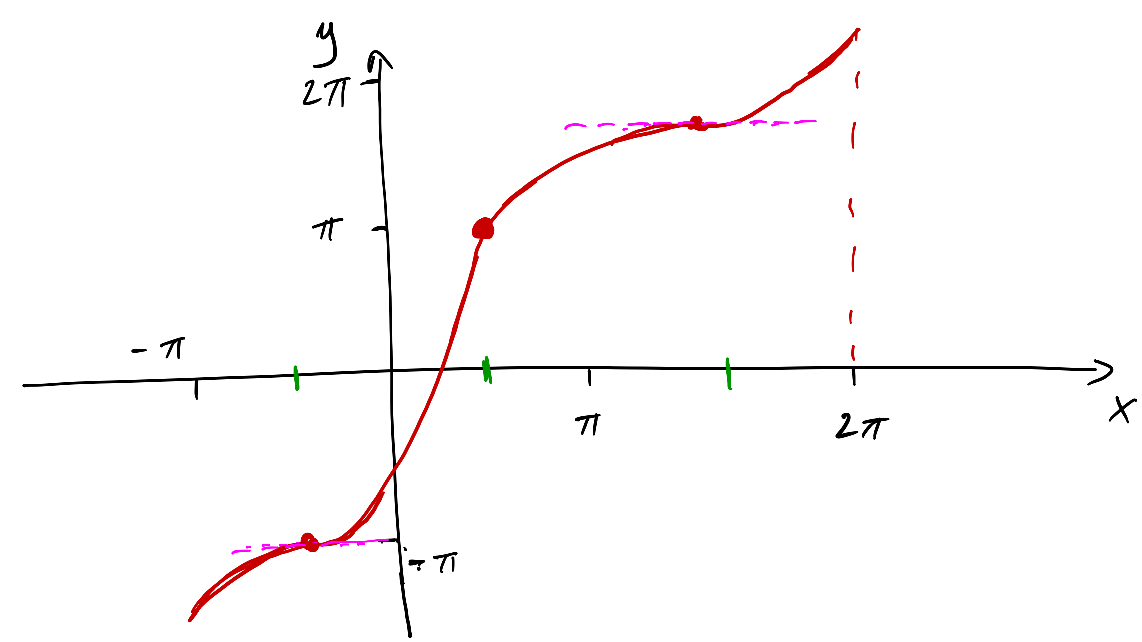
answer

f is concave up on $(-\frac{\pi}{2}, \frac{\pi}{2})$ and $(\frac{3\pi}{2}, 2\pi)$. f is concave down on $(-\pi, -\frac{\pi}{2}) \cup (\frac{\pi}{2}, \frac{3\pi}{2})$.

(d) Find the inflection points.

Inflection pts: $(-\frac{\pi}{2}, -\pi), (\frac{\pi}{2}, \pi), (\frac{3\pi}{2}, 3\pi)$

(e) Sketch the graph.



8. Evaluate the following limits. Show your work.

(a) $$\underbrace{\lim_{x\to 0}\frac{x-e^x+1}{\sin x}}_{\text{form } \frac{0}{0}} \overset{H}{=} \lim_{x\to 0}\frac{1-e^x}{\cos x}$$

$$= \frac{0}{1} = 0$$

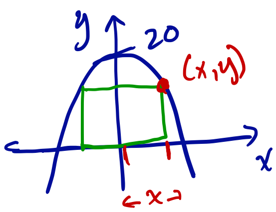
(b) $$\overbrace{\lim_{x\to 0^+}(1+2x)^{1/x}}^{\text{form } 1^\infty} = \boxed{e^2}$$

Let $y=(1+2x)^{1/x}$. So $\ln y = \frac{1}{x}\ln(1+2x) = \frac{\ln(1+2x)}{x}$

$$\underbrace{\lim_{x\to 0^+}\frac{\ln(1+2x)}{x}}_{\text{form } \frac{0}{0}} \overset{H}{=} \lim_{x\to 0^+}\frac{\frac{2}{1+2x}}{1} = \lim_{x\to 0^+}\frac{2}{1+2x} = 2$$

9. Find the rectangle of maximum area that can be inscribed inside the region bounded above by $y=20-x^2$ and bounded below by the x-axis. (Assume the base of the rectangle lies on the x-axis.)

Sketch



Construct function:

area $= A = 2xy$, $y = 20-x^2$.

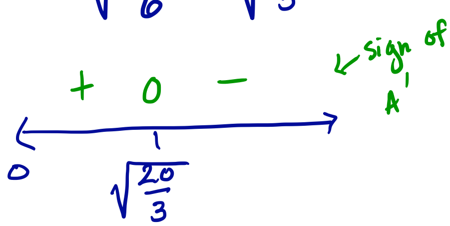
So $A(x) = 2x(20-x^2) = 40x - 2x^3$ on $(0,\infty)$

We want to maximize A.

Apply First Derivative Test:

$A'(x) = 40 - 6x^2 = 0$

$x = \sqrt{\frac{40}{6}} = \sqrt{\frac{20}{3}}$



So $A(x)$ has an absolute maximum at $x=\sqrt{\frac{20}{3}}$.

(The change in sign of A' tells us $x=\sqrt{\frac{20}{3}}$ corresponds to a local maximum. This local maximum must be absolute, on $(0,\infty)$ because $A(x)$ has exactly one critical point on the interval $(0,\infty)$.

Answering the question:

The rectangle w/ maximum area with base on x-axis has upper right corner at point

$$\left(\sqrt{\frac{20}{3}},\ \frac{40}{3}\right).$$