LECTURE: 5-2 THE DEFINITE INTEGRAL

Example 1: Estimate the area under $f(x) = x^2 - 2x$ on $[0, 4]$ with $n = 8$ using the left Riemann sum.

Some helpful sums:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} c = nc$$

$$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$$

$$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

Example 2: Find the area under $f(x) = x^2 - 2x$ on $[0, 4]$ exactly.

The Midpoint Rule:

Example 3: Use the midpoint rule with $n = 5$ to approximate the area under $f(x) = \dfrac{1}{x}$ on $[1, 2]$.

Definition of a Definite Integral If f is a function defined for $a \leq x \leq b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0(a), x_1, x_2, \cdots, x_n(b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \cdots, x_n^*$ be **sample points** in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the **definite integral of f from a to b** is

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

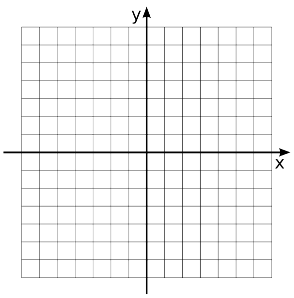
Provided this limit exists and gives the same value or all possible choices of sample points. If it does exist, we say that f is **integrable** on $[a, b]$.

Theorem If f is continuous on $[a, b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a, b]$; that is, the definite integral $\int_a^b f(x)dx$ exists.

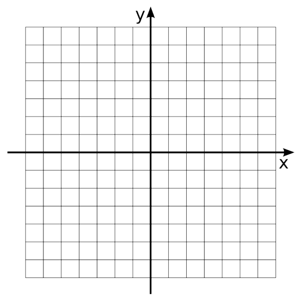
The thing to remember is that a definite integral represents the *signed* area under a curve. If a curve is above the x-axis that area is ______________, if the curve is below the x-axis the area is ______________. Some definite integrals can be found by graphing the curve and using the areas of known geometric shapes to then find the value of the definite integral.

Example 4: Evaluate the following integrals by interpreting each in terms of areas.

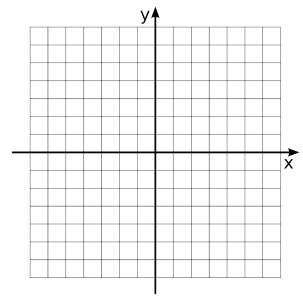
a) $\int_0^3 (x-1)dx$



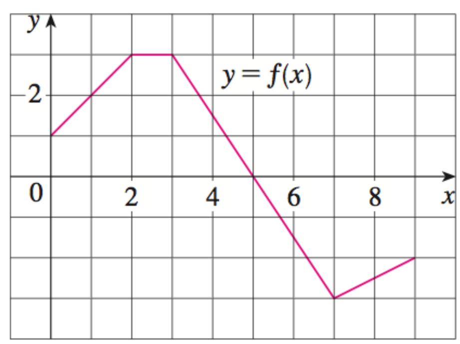
b) $\int_0^4 \sqrt{16-x^2}dx.$



c) $\int_{-3}^3 (2+\sqrt{9-x^2})dx$



Example 5: The graph of f is shown. Evaluate each integral by interpreting it in terms of areas.



(a) $\int_2^5 f(x)dx$

(b) $\int_5^9 f(x)dx$

(c) $\int_3^7 f(x)dx$

Properties of the Definite Integral:

1. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

2. $\int_a^a f(x)dx = 0$

3. $\int_a^b c\,dx = c(b-a)$

4. $\int_a^b cf(x)dx = c\int_a^b f(x)dx$

5. $\int_a^b [f(x)\pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

Example 6: Using the fact that $\int_0^1 x^2 dx = \frac{1}{3}$, evaluate the following using the properties of integrals.

(a) $\int_1^0 t^2 dt$

(b) $\int_0^1 (4 + 3x^2) dx$.

Example 7: If it is known that $\int_0^{10} f(x)dx = 17$ and $\int_0^8 f(x)dx = 12$, find $\int_8^{10} f(x)dx$.

Example 8: Evaluate $\int_3^3 x \sin x dx$.

Comparison Properties of the Integral

- If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq 0$
- If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.
- If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$

Example 9: Use the final property given above to estimate the value of the integral $\int_0^1 x^4 \, dx$.