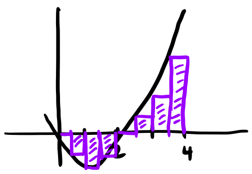
LECTURE: 5-2 THE DEFINITE INTEGRAL

Example 1: Estimate the area under $f(x) = x^2 - 2x$ on $[0, 4]$ with $n = 8$ using the

$= x(x-2)$

(a) Left Riemann Sum



$L_8 = \frac{1}{2}\left(f(0) + f(1/2) + f(1) + f(3/2) + f(2) + f(5/2) + f(3) + f(7/2) \right)$

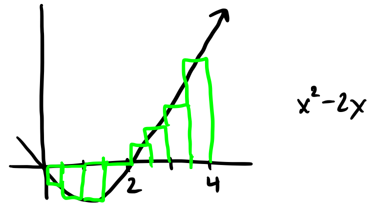
$= \frac{1}{2}\left(0 + (\frac{1}{4} - 1) + (1-2) + (\frac{9}{4} - 3) + (4-4) + (\frac{25}{4} - 5) + (9-6) + (\frac{49}{4} - 7)\right)$

$= \frac{1}{2}\left(\frac{84}{4} - 14\right)$

$= \frac{1}{2}(21 - 14)$

$= 7/2$

(b) Right Riemann Sum



$R_8 = \frac{1}{2}\left(f(1/2) + f(1) + f(3/2) + f(2) + f(5/2) + f(3) + f(7/2) + f(4) \right)$

$= \frac{1}{2}\left((\frac{1}{4} - 1) + (1-2) + (9/4 - 3) + (4-4) + (\frac{25}{4} - 5) + (\frac{49}{4} - 7) + (16-8)\right)$

$= \frac{1}{2}\left(\frac{84}{4} - 1 + 1 - 3 - 5 - 7 + 8\right)$

$= \frac{1}{2}(21 - 7)$

$= \frac{14}{2}$

$= \boxed{7}$

Example 2: Find $\int_0^4 (x^2 - 2x)dx$ exactly.

$\boxed{\Delta x = \frac{b-a}{n} = \frac{4}{n}}$

$0 \quad 4/n \quad 8/n \quad 12/n \ldots \quad 4 \cdot \frac{n}{n}$

$R_n = \frac{4}{n}\left(\left(\frac{4}{n}\right)^2 - 2\left(\frac{4}{n}\right) + \left(\frac{4 \cdot 2}{n}\right)^2 - 2\left(\frac{4 \cdot 2}{n}\right) + \ldots \quad \left(4 \cdot \frac{n}{n}\right)^2 - 2\left(4 \cdot \frac{n}{n}\right) \right)$

$= \frac{4}{n} \sum_{i=1}^{n} \left(\left(\frac{4i}{n}\right)^2 - \frac{8i}{n} \right)$

$= \frac{4}{n} \left(\frac{16}{n^2} \sum_{i=1}^{n} i^2 - \frac{8}{n} \sum_{i=1}^{n} i \right)$

$= \frac{4}{n} \left(\frac{16}{n^2} \left(\frac{n(n+1)(2n+1)}{6} \right) - \frac{8}{n} \left(\frac{n(n+1)}{2} \right) \right)$

$= \frac{4}{n} \left(\frac{8(2n^2 + 3n + 1)}{3n} - 4(n+1) \right)$

$= \frac{32(2n^2 + 3n + 1)}{3n^2} - \frac{16(n+1)}{n}$

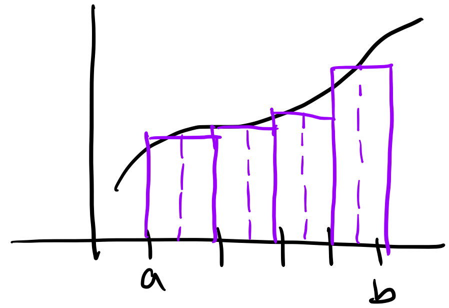
$A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} \left(\frac{64n^2 + 96n + 1}{3n^2} + \frac{(-16n - 16)}{n} \right)$

$= 64/3 - 16 = 64/3 - 16(3/3)$

$= 64/3 - 48/3 = \boxed{16/3}$

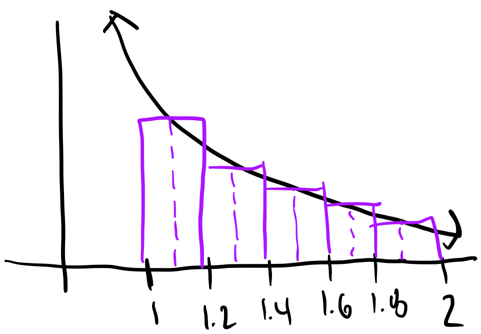
The Midpoint Rule:

Idea → approximate height of a function w/ f(mid) in a sub-interval.



Example 3: Use the midpoint rule with $n = 5$ to approximate $\int_1^2 \frac{1}{x}dx$; $f(x) = 1/x$

$\Delta x = (2-1)/5 = 1/5 = 0.2$



$$M_5 = \frac{1}{5}\left(\frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{1.5} + \frac{1}{1.7} + \frac{1}{1.9}\right)$$

$$\approx \boxed{0.692}$$

Definition of a Definite Integral If f is a function defined for $a \le x \le b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b-a)/n$. We let $x_0(a), x_1, x_2, \cdots, x_n(b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \cdots, x_n^*$ be **sample points** in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the **definite integral of** f **from** a **to** b is

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

Provided this limit exists and gives the same value or all possible choices of sample points. If it does exist, we say that f is **integrable** on $[a, b]$.

This is a very technical definition that we will almost never use.

these can be left, right, or midpoints.. The result will be the same.

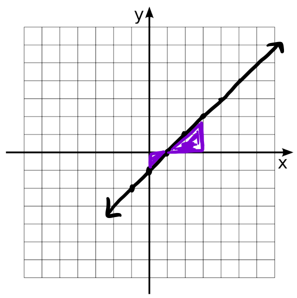
Theorem If f is continuous on $[a, b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a, b]$; that is, the definite integral $\int_a^b f(x)dx$ exists.

check these two conditions. If they are true, then you can find the integral // area under curve.

The thing to remember is that a definite integral represents the *signed* area under a curve. If a curve is above the x-axis that area is positive, if the curve is below the x-axis the area is negative. Some definite integrals can be found by graphing the curve and using the areas of known geometric shapes to then find the value of the definite integral.

Example 4: Evaluate the following integrals by interpreting each in terms of areas.

a) $\int_0^3 (x-1)dx$



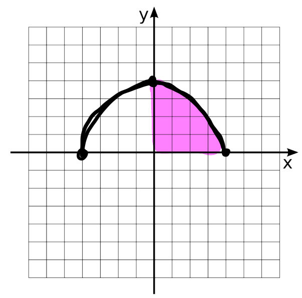
$A = -\frac{1}{2}(1)(1) + \frac{1}{2}(2)(2)$

$= -\frac{1}{2} + \frac{4}{2}$

$= \boxed{\frac{3}{2}}$

$= \boxed{1.5}$

b) $\int_0^4 \sqrt{16-x^2}dx.$



note $y=\sqrt{16-x^2} \Rightarrow y^2 = 16-x^2$

$\Rightarrow x^2+y^2=16$

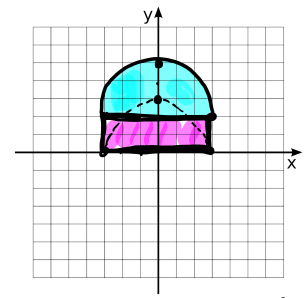
this is the upper ½ of this circle.

$A = \frac{1}{4}\pi r^2 = \frac{1}{4}\pi(4^2)$

$= \boxed{4\pi}$

c) $\int_{-3}^3 (2+\sqrt{9-x^2})dx$

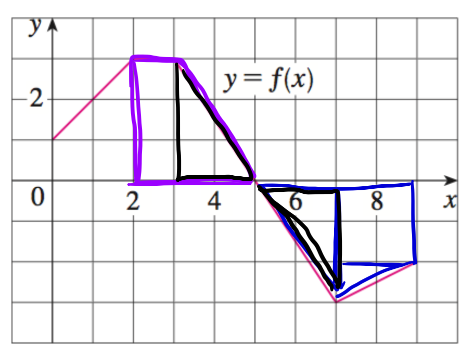
This is a semi-circle w/ r = 3 shifed up 2!



$A = 6(2) + \frac{1}{2}\pi \cdot 3^2$

$= \boxed{12 + \frac{9\pi}{2}}$

Example 5: The graph of f is shown. Evaluate each integral by interpreting it in terms of areas.



(a) $\int_2^5 f(x)dx = 1 \cdot 3 + \frac{1}{2}(2)(3)$

$= 6$

(b) $\int_5^9 f(x)dx = -\left(\frac{1}{2}(2)(3) + 2\cdot 2 + \frac{1}{2}(1)(2)\right)$

$= -(3+4+1) = \boxed{-8}$

(c) $\int_3^7 f(x)dx = \boxed{0}$

Properties of the Definite Integral:

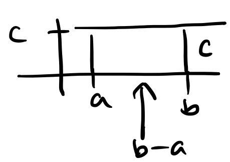
1. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

   $\Delta x = \frac{b-a}{n}$, $\Delta x = \frac{a-b}{n} = \frac{-(b-a)}{n}$

2. $\int_a^a f(x)dx = 0$ area of a line is zero!

3. $\int_a^b c\,dx = c(b-a)$

   rectangle!



4. $\int_a^b cf(x)dx = c\int_a^b f(x)dx$

5. $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

Example 6: Using the fact that $\int_0^1 x^2 dx = \frac{1}{3}$, evaluate the following using the properties of integrals.

(a) $\int_1^0 t^2 dt = -\int_0^1 t^2 dt$

$= \boxed{-1/3}$

(b) $\int_0^1 (4 + 3x^2) dx = \int_0^1 4\,dx + 3\int_0^1 x^2\,dx$

$= 4(1-0) + 3 \cdot 1/3$

$= 4 + 1$

$= \boxed{5}$

Example 7: If it is known that $\int_0^{10} f(x)dx = 17$ and $\int_0^8 f(x)dx = 12$, find $\int_8^{10} f(x)dx$.

$\int_0^{10} f(x)\,dx - \int_0^8 f(x)\,dx = \int_8^{10} f(x)\,dx$

$17 - 12 = \int_8^{10} f(x)\,dx \Rightarrow \boxed{\int_8^{10} f(x)\,dx = 5}$

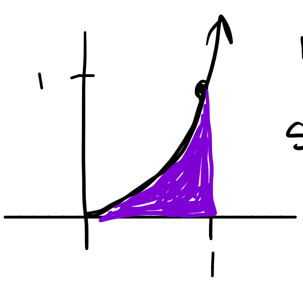
Example 8: Evaluate $\int_3^3 x \sin x dx$. $= \boxed{0}$

Comparison Properties of the Integral

- If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq 0$
- If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.
- If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$

Example 9: Use the final property given above to estimate the value of the integral.

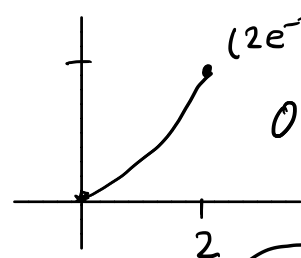
(a) $\int_0^1 x^4 dx \leftarrow$ $f(x) = x^4$



note $0 \leq f(x) \leq 1$

so $\boxed{0 \leq \int_0^1 x^4\,dx \leq 1}$

(b) $\int_0^2 xe^{-x} dx$ $f(x) = xe^{-x}$



$(2e^{-2})$ note $0 \leq f(x) \leq 2e^{-2}$

$0\,(2) \leq \int_0^2 xe^{-x}dx \leq 2 \cdot 2e^{-2}$

$\boxed{0 \leq \int_0^2 xe^{-x}\,dx \leq 4/e^2}$