# Lecture: 5-5 The Substitution Rule (Part 1)

**Example 1:** How would we factor $x^4 - 5x^2 + 6$ and how might it relate to finding $\int 2x\sqrt{1+x^2}\,dx$?

$u = x^2 \qquad u^2 - 5u + 6$

$= (u-2)(u-3)$

$= (x^2-2)(x^2-3)$

$u = 1 + x^2$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$\int 2x\sqrt{1+x^2}\,dx$

$= \int \sqrt{u}\,du$

$= \frac{2}{3}u^{3/2} + C$

$\boxed{= \frac{2}{3}(1+x^2)^{3/2} + C}$

> **The Substitution Rule** If $u = g(x)$ is a differentiable function whose range is an interval $I$ and $f$ is continuous on $I$ then
>
> $$\int f(g(x))g'(x)\,dx = \int f(u)du$$
>
> Note, the substitution rule is basically undoing the chain rule.

So ... $u = g(x)$

$\frac{du}{dx} = g'(x) \Rightarrow du = g'(x)dx$

**Example 2:** Evaluate $\int x^3 \cos(x^4+2)\,dx$ two different ways:

(a) solve for $dx$.

$u = x^4 + 2$

$du = 4x^3\,dx$

$dx = \frac{du}{4x^3}$

$= \int x^3 \cos(u) \frac{du}{4x^3}$

$= \frac{1}{4}\int \cos u\,du$

$= \frac{1}{4}\sin u + C$

$\boxed{= \frac{1}{4}\sin(x^4+2) + C}$

(b) solve for $x^3\,dx$.

$u = x^4 + 2$

$du = 4x^3\,dx$

$\frac{1}{4}du = x^3\,dx$

$= \frac{1}{4}\int \cos(u)\,du$

$= \frac{1}{4}\sin(x^4+2) + C$

The trickiest thing about subsitution is deciding what to substitute. As substiution is (usually) undoing the chain rule you chould let your $u$ be the inside function. Choose $u$ to be the stuff inside of a power, root sign, denominator, or trigonometric function. When you are choosing your $u$ the derivative of $u$ should appear elsewhere in the integrand up to a constant multiple. The only way to get better is a lot of practice!
Once you make your subsitution the integral usually simplifies considerably. If your original variable does not completely disappear when making the subsitution you either (a) chose a subsitution that doesn't work or (b) made a mistake. At this stage you can try something different, or start your original substitution again.

**Example 3:** Evaluate the following indefinite integrals.

(a) $\int \sqrt{3x+2}\,dx$

$u = 3x+2$

$du = 3dx$

$\frac{du}{3} = dx$

$= \frac{1}{3}\int u^{1/2}\,du$

$= \frac{1}{3}\cdot\frac{2}{3}u^{3/2} + C$

$\boxed{= \frac{2}{9}(3x+2)^{3/2} + C}$

(b) $\int \cos^4 x \sin x\,dx$

$(\cos x)^4$

$u = \cos x$

$du = -\sin x\,dx$

$-du = \sin x\,dx$

$= -\int u^4\,du$

$= -\frac{1}{5}u^5 + C$

$\boxed{= \frac{-1}{5}\cos^5 x + C}$

**Example 4:** Evaluate the following indefinite integrals.

(a) $\int \frac{\sec^2 x}{\tan^2 x}\,dx$

$u = \tan x$

$du = \sec^2 x\,dx$

$= \int \frac{1}{u^2}\,du$

$= \int u^{-2}\,du$

$= -u^{-1} + C$

$\boxed{= -\frac{1}{\tan x} + C}$

(b) $\int \frac{x}{\sqrt{1-x^4}}\,dx$

$u = x^2$

$du = 2x\,dx$

$\frac{du}{2} = x\,dx$

$= \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}\,du$

$= \frac{1}{2}\sin^{-1} u + C$

$\boxed{= \frac{1}{2}\sin^{-1}(x^2) + C}$

**Example 5:** Evaluate the following indefinite integrals.

(a) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

$u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$2\,du = \frac{1}{\sqrt{x}}\,dx$

$= 2\int e^u\,du$

$= 2e^u + C$

$\boxed{= 2e^{\sqrt{x}} + C}$

(b) $\int \frac{\arctan x}{x^2+1}\,dx$

$u = \arctan x$

$du = \frac{1}{x^2+1}\,dx$

$= \int u\,du$

$= \frac{1}{2}u^2 + C$

$\boxed{= \frac{1}{2}(\arctan x)^2 + C}$

**Example 6:** Evaluate the following indefinite integrals.

(a) $\displaystyle\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$

$u = \sin\theta$

$du = \cos\theta\, d\theta$

$= \int u^{-2}\,du$

$= -u^{-1} + C$

$= -\dfrac{1}{\sin\theta} + C$

(b) $\displaystyle\int \tan x\,dx$

$= \displaystyle\int \frac{\sin x}{\cos x}\,dx$

$u = \cos x$

$du = -\sin x\,dx$

$= -\displaystyle\int \frac{du}{u}$

$= -\ln|u| + C$

$\boxed{= -\ln|\cos x| + C \text{ or } \ln|\sec x| + C}$

**Example 7:** Evaluate the following indefinite integrals.

(a) $\displaystyle\int (1+\tan x)^5 \sec^2 x\,dx$

$u = 1+\tan x$

$du = \sec^2 x\,dx$

$= \int u^5\,du$

$= \dfrac{1}{6}u^6 + C$

$\boxed{= \dfrac{1}{6}(1+\tan x)^6 + C}$

(b) $\displaystyle\int \frac{\cos(\pi/x)}{x^2}\,dx$

$u = \pi/x$

$du = -\dfrac{\pi}{x^2}\,dx$

$-\dfrac{1}{\pi}\,du = \dfrac{1}{x^2}\,dx$

$= -\dfrac{1}{\pi}\displaystyle\int \cos u\,du$

$= -\dfrac{1}{\pi}\sin u + C$

$\boxed{= -\dfrac{1}{\pi}\sin(\pi/x) + C}$

**Example 8:** Evaluate $\displaystyle\int \frac{5+x}{1+x^2}\,dx.$ $= \displaystyle\int \frac{5}{1+x^2}\,dx + \int \frac{x}{1+x^2}\,dx$

$u = 1+x^2$

$du = 2x\,dx$

$\dfrac{du}{2} = x\,dx$

$= 5\arctan x + \dfrac{1}{2}\displaystyle\int \frac{1}{u}\,du$

$\boxed{= 5\arctan x + \dfrac{1}{2}\ln|1+x^2| + C}$

Sometimes when you do substitution you also end up solving for your variable in the substitution. For example:

**Example 9:** Evaluate $\int x^5\sqrt{x^3+1}\,dx$. $= \int x^3 \cdot x^2 \sqrt{x^3+1}\,dx$

$u = x^3+1$

$du = 3x^2\,dx$

$\frac{1}{3}du = x^2 dx$

$x^3 = u-1$

$\int (u-1)\cdot u^{1/2}\cdot \frac{1}{3}du$

$= \frac{1}{3}\int (u^{3/2} - u^{1/2})\,du$

$= \frac{1}{3}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + C$ ← plug in $x^3+1$ for $u$.

**Example 10:** Evaluate $\int x\sqrt{x+2}\,dx$

$u = x+2$

$du = dx$

$x = u-2$

$= \int (u-2)\sqrt{u}\,du$

$= \int \left(u^{3/2} - 2u^{1/2}\right)du$

$= \frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2} + C$

$\boxed{= \frac{2}{5}(x+2)^{5/2} - \frac{4}{3}(x+2)^{3/2} + C}$

## Definite Integrals

**The Substitution Rule for Definite Integrals:** If $g'$ is continuous on $[a, b]$ and $f$ is continuous on the range of $u = g(x)$, then

plug in lower and upper bounds to change!

$$\int_{x=a}^{x=b} f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

**Example 11:** Evaluate $\int_0^{\pi/2} \sin^3 x \cos x\,dx$ two ways:

$u = \sin x$

$du = \cos x\,dx$

$0$: $u = \sin 0 = 0$

$u = \sin \pi/2 = 1$

a) going back to $x$'s

$\int_0^{\pi/2} \sin^3 x \cos x\,dx = \int_{x=0}^{x=\pi/2} u^3\,du$

$= \frac{1}{4}u^4 \Big|_{x=0}^{x=\pi/2}$

$= \frac{1}{4}\sin^4 x\Big|_0^{\pi/2}$

$= \frac{1}{4}\left(\sin^4 \frac{\pi}{2} - \sin^4 0\right) = \frac{1}{4}$

b) using substitution

$\int_0^{\pi/2} \sin^3 x \cos x\,dx = \int_0^1 u^3\,du$

$= \frac{1}{4}u^4\Big|_0^1$

$= \frac{1}{4}(1^4 - 0^4)$

$\boxed{= 1/4}$