LECTURE: 5-5 THE SUBSTITUTION RULE (PART 2)

Recall:

> **The Substitution Rule for Definite Integrals:** If g' is continuous on $[a,b]$ and f is continuous on the range of $u = g(x)$, then
> $$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Example 1: Evaluate the following definite integrals.

a) $\displaystyle\int_e^{e^3} \frac{1}{x(\ln x)^2}\,dx = \int_1^3 \frac{1}{u^2}\,du$

$u = \ln x$

$du = \frac{1}{x}dx$

$u = \ln e^3 = 3$

$u = \ln e = 1$

$= -\frac{1}{u}\Big|_1^3$

$= -\frac{1}{3} - \left(-\frac{1}{1}\right)$

$\boxed{= 2/3}$

b) $\displaystyle\int_1^2 x\sqrt{x-1}\,dx = \int_0^1 (u+1)\sqrt{u}\,du$

$u = x-1$

$du = dx$

$x = u+1$

$u = 2-1 = 1$

$u = 1-1 = 0$

$= \int_0^1 \left(u^{3/2} + u^{1/2}\right) du$

$= \frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2}\Big|_0^1$

$= \frac{2}{5} + \frac{2}{3} - 0$

$\boxed{= 16/15}$

Example 2: Evaluate the following definite integrals.

a) $\displaystyle\int_0^2 \frac{x}{x^2+4}\,dx = \frac{1}{2}\int_4^8 \frac{1}{u}\,du$

$u = x^2+4$

$du = 2x\,dx$

$u_2 = 2^2+4 = 8$

$u_1 = 0^2+4 = 4$

$= \frac{1}{2}\ln|u|\Big|_4^8$

$= \frac{1}{2}(\ln 8 - \ln 4)$

$\boxed{= \frac{1}{2}\ln 2}$

b) $\displaystyle\int_a^b z^2\cos(1-z^3)\,dz = -\frac{1}{3}\int_{1-a^3}^{1-b^3} \cos u\,du$

$u = 1-z^3$

$du = -3z^2\,dz$

$= -\frac{1}{3}\sin u\Big|_{1-a^3}^{1-b^3}$

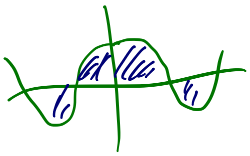
$\boxed{= -\frac{1}{3}\sin(1-b^3) + \frac{1}{3}\sin(1-a^3)}$

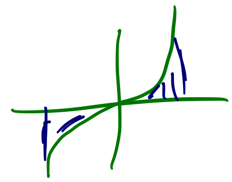
Symmetry

> - A function f is even if $f(-a) = f(a)$. Even functions are symmetric about the y-axis.
> - A function f is odd if $f(-a) = -f(a)$. Odd functions are symmetric about the origin.

Integrals of Even/Odd Functions: Suppose a function $f(x)$ is (blank) on $[-a, a]$. Then,

(a) (even) $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$ (b) (odd) $\int_{-a}^{a} f(x)\,dx = \underline{0}$!





Example 3: Evaluate the following definite integrals.

(a) $\int_{-2}^{2} (x^2+1)\,dx$ $2\int_0^2 (x^2+1)\,dx$

$f(x) = x^2+1$

$f(-x) = (-x)^2+1$

$= x^2+1$

even!

$2\left(\frac{1}{3}x^3 + x\right)\Big|_0^2$

$2\left(\frac{1}{3}(8) + 2 - 0\right)$

$\frac{16}{3} + \frac{12}{3} = \boxed{\frac{28}{3}}$

(b) $\int_{-1}^{1} \frac{\tan x}{1+x^2}\,dx$ $= 0$

$f(-x) = \frac{\tan(-x)}{1+(-x)^2}$

$= -\frac{\tan x}{1+x^2}$

odd

Example 4: If f is continuous and $\int_0^9 f(x)\,dx = 4$, find $\int_0^3 x f(x^2)\,dx$.

$\int_0^3 x f(x^2)\,dx = \frac{1}{2}\int_0^9 f(u)\,du$

$u = x^2$ $\quad u_1 = 0^2 = 0$

$du = 2x\,dx$ $\quad u_2 = 3^2 = 9$

$= \frac{1}{2}(4) = \boxed{2}$

Example 5: Evaluate $\int_{-3}^{3} (x+5)\sqrt{9-x^2}\,dx$.

$= \int_{-3}^{3} x\sqrt{9-x^2}\,dx + 5\int_{-3}^{3}\sqrt{9-x^2}\,dx$

odd!

$= 0 + 5 \cdot \frac{1}{2}(\pi)(3)^2 = \boxed{\frac{45\pi}{2}}$

Example 6: Doing (some) substitutions quickly. In later Calculus courses (Calculus 2 especially) it is quite useful to be able to do some very simple substitutions without having to go through writing out u and du. Do the following integrals using substitution and then see if you can see the pattern well enough to not need to do all of the work.

(a) $\int e^{5x}\,dx$

$u = 5x$, $du = 5dx$, $\frac{1}{5}du = dx$

$= \frac{1}{5}\int e^u\,du$

$\boxed{= \frac{1}{5}e^{5x} + C}$

(b) $\int \sin\left(\frac{\pi}{2}x\right)\,dx$

$u = \frac{\pi}{2}x$, $du = \frac{\pi}{2}dx$

$= \frac{2}{\pi}\int \sin u\,du$

$= -\frac{2}{\pi}\cos u + C$

$\boxed{= -\frac{2}{\pi}\cos\left(\frac{\pi}{2}x\right) + C}$

(c) $\int \sqrt{1-2x}\,dx$

$u = 1-2x$, $du = -2dx$

$= -2\int u^{1/2}\,du$

$= -2\left(\frac{2}{3}\right)u^{3/2} + C$

$\boxed{= -\frac{4}{3}(1-2x)^{3/2} + C}$

Example 7: Integrate the following functions. Check your answers using a derivative.

(a) $\int \sec^2\left(\frac{\pi}{4}\theta\right)\,d\theta$

$= \frac{4}{\pi}\tan\left(\frac{\pi}{4}\theta\right) + C$

(b) $\int \sec(2x)\tan(2x)\,dx$

$= \frac{1}{2}\sec(2x) + C$

(c) $\int \sqrt{1+4x}\,dx$

$= \frac{2}{3}\cdot\frac{1}{4}(1+4x)^{3/2} + C$

$= \frac{1}{6}(1+4x)^{3/2} + C$

Example 8: Evaluate the following integrals.

(a) $\int xe^{-x^2}\,dx$

$u = -x^2$, $du = -2x\,dx$, $-\frac{1}{2}du = x\,dx$

$= -\frac{1}{2}\int e^u\,du$

$= -\frac{1}{2}e^u + C$

$\boxed{= -\frac{1}{2}e^{-x^2} + C}$

(b) $\int_1^4 \frac{1}{x^2}\sqrt{1+\frac{1}{x}}\,dx$

$u = 1 + \frac{1}{x}$, $du = -\frac{1}{x^2}dx$

$u_1 = 1 + \frac{1}{1} = 2$, $u_2 = 1 + \frac{1}{4} = 5/4$

$= -\int_2^{5/4} u^{1/2}\,du$

$= -\frac{2}{3}u^{3/2}\Big|_2^{5/4}$

$= -\frac{2}{3}\left(\left(5/4\right)^{3/2} - 2^{3/2}\right)$

$\boxed{= -2/3\left(\frac{5\sqrt{5}}{8} - 2\sqrt{2}\right)}$