Final Review - Chapter 4
(Applications of Differentiation)

- Find critical numbers of a function.
- Find the absolute maximum and absolute minimum of a function on a closed interval.

Example 1: Find the absolute maximum and minimum of $f(x) = xe^{x/2}$ on $[-3, 1]$

$$f'(x) = e^{x/2} + \frac{1}{2}xe^{x/2} = 0$$

$$e^{x/2}\left(1 + \tfrac{1}{2}x\right) = 0$$

$\frac{1}{2}x = -1$ $\quad \underline{x = -2}$ crit. point.

x	-3	-2	1
$f(x)$	$-3e^{-3/2}$	$-2e^{-2/2}$	$e^{1/2}$
	-0.67	$-.73$	1.65
		min	max

abs min -0.73 @ $x = -2$

abs max 1.65 @ $x = 1$

- Determine where a function is increasing decreasing.
- Determine where a function is concave up and concave down.

Example 2: Given $G(x) = 5x^{2/3} - 2x^{5/3}$

(a) Find the intervals of increase/ decrease.

$$G'(x) = \frac{10}{3}x^{-1/3} - \frac{10}{3}x^{2/3} = 0 \qquad \left(* \frac{3}{10}x^{-1/3}\right)$$

$$\frac{10}{3}x^{-1/3}(1 - x) = 0$$

$$\frac{10(1-x)}{3x^{1/3}} = 0$$

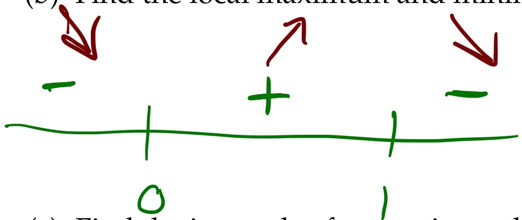
$\boxed{x = 0, 1 \text{ c.p.}}$

$-$ ↘ $\quad$ 0 $\quad$ $+$ ↗ $\quad$ 1 $\quad$ $-$ ↘

decrease: $(-\infty, 0) \cup (1, \infty)$

increase: $(0, 1)$

(b) Find the local maximum and minimum values.



$\Rightarrow$ local min @ $x = 0$ ← $f(0) = 0$

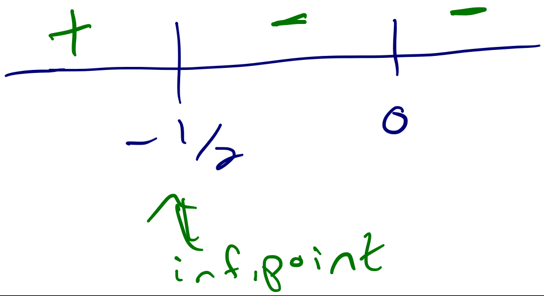
" max @ $x = 1$ ← $f(1) = 3$

(c) Find the intervals of concavity and the inflection points.

$$G''(x) = -\frac{10}{9}x^{-4/3} - \frac{20}{9}x^{-1/3} = 0$$

$$-\frac{10}{9}x^{-4/3}(1 + 2x) = 0$$

$x = 0, -1/2$ - possible inf points



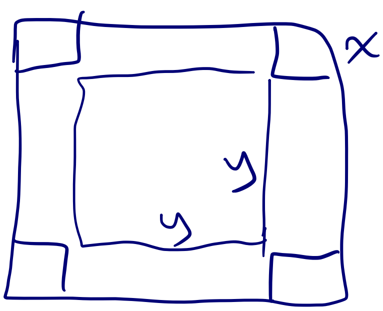
concave up: $(-\infty, -1/2)$

concave down: $(-1/2, \infty)$

inf. point: $(-1/2, G(-1/2))$

- Solve max/ min optimization problems.

Example 3: A box with an open top is to be constructed from a square piece of cardboard, 3 ft wide, by cutting a square from each of the four corners and bending up the sides. Find the largest volume that such a box can have.

① 

② $V = xy^2$

$3 = 2x + y$

$y = 3 - 2x$

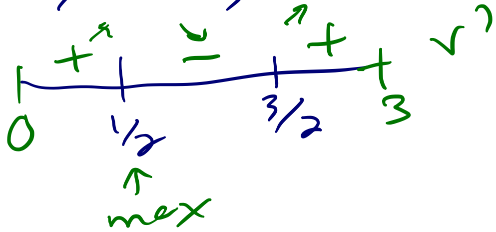
④ $V = x(3 - 2x)^2$

$V = 9x - 12x^2 + 4x^3$

$V' = 9 - 24x + 12x^2 = 0$

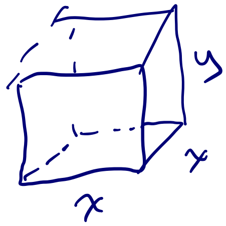
$(x - 1/2)(x - 3/2) = 0$

$x = 1/2, 3/2$ ft



$x = 1/2 \Rightarrow \boxed{V = 2\text{ ft}^3}$

Example 4: Suppose a box with a square base and open top must have a volume of 32 m^3. Find the dimensions of the box that minimize the amount of material used.



$S = x^2 + 4xy$

$V = x^2 y = 32$

$y = \frac{32}{x^2}$

$S = x^2 + 4x\left(\frac{32}{x^2}\right)$

$S = x^2 + \frac{128}{x}$

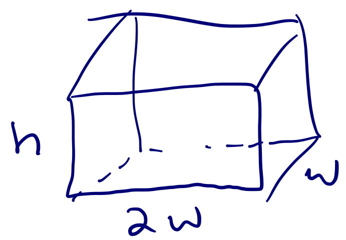
$S' = 2x - \frac{128}{x^2} = 0 \Rightarrow x^3 = 64$, $x = 4$

S': $-$ | $+$ at 4

minimum!

$\boxed{\begin{array}{l} y = 2 \text{ m} \\ x = 4 \text{ m} \end{array}}$

Example 5: A rectangular storage container with an open top is to have a volume of 10 m^3. The length of the base is twice the width. Material for the base costs \$10 per square meter. Material for the sides costs \$ 6 per square meter. Find the costs of materials for the cheapest such container.



cost $C = 10(w)(2w) + 2(6)(h)(w) + 2(6)(h)(2w)$

$C(w) = 20w^2 + 60w^{-1} + 120w^{-1}$

$C(w) = 20w^2 + 180w^{-1}$

$C'(w) = 40w - 180w^{-2} = 0$

$\Rightarrow w = \sqrt[3]{9/2}$ - c.p

$V = 2w \cdot w \cdot h = 10$

$h = \frac{5}{w^2}$

$-$ | $+$ at $\sqrt[3]{9/2}$ $\leftarrow$ location of min!

$\boxed{C(\sqrt[3]{9/2}) = 20(9/2)^{2/3} + 180/\sqrt[3]{9/2}}$

- Apply Newton's method to take a "step" (get a better approximation of a root of a function.)

Example 6: Use one iteration of Newton's method with $x_1 = -1$ to get a better approximation of the root of $f(x) = x^7 + 4$. [I.e., find x_2.] After that, graph $f(x)$ and demonstrate how x_2 was obtained from x_1.

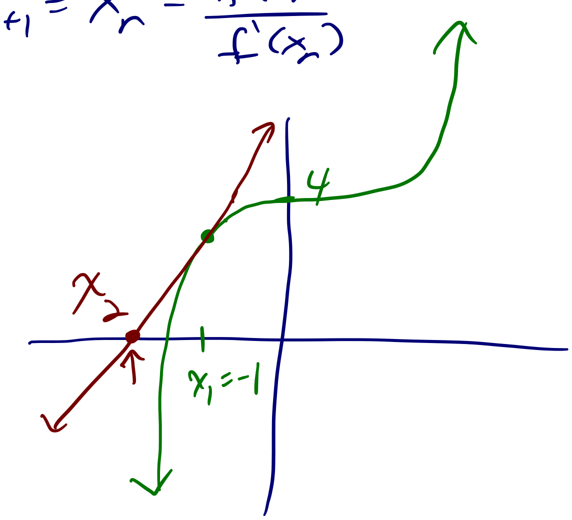
$f'(x) = 7x^6$

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$x_2 = -1 - \frac{(-1)^7 + 4}{7(-1)^6}$

$= -1 - \frac{3}{7}$

$\boxed{= -10/7}$



Final Review - Chapter 5
(Integration)

Example 1: Find the most general antiderivative of the function.

a) $g(x) = \frac{1}{x} + \frac{1}{x^2+1}$

b) $f(x) = \frac{x^2+\sqrt{x}}{x}$ $= x + x^{-1/2}$

$G(x) = \ln|x| + \arctan x + C$

$F(x) = \frac{1}{2}x^2 + 2\sqrt{x} + C$

Example 2: A particle is moving with $v(t) = 2t - 1/(1+t^2)$ and $s(0) = 1$. Find the position of the particle.

$v(t) = \frac{2t}{1+t^2} - \frac{1}{1+t^2}$

$s(t) = \ln(1+t^2) - \arctan x + C$

$s(0) = \ln(1) - \arctan 0 + C = 1 \Rightarrow C = 1$

$\boxed{s(t) = \ln(1+t^2) - \arctan x + C}$

Example 3: Compare/contrast the applications of FTC below.

a) Find the derivative of
$g(x) = \int_1^{x^2} t^3\sqrt{1+t^4}\,dt$

b) Evaluate $\int_1^a t^3\sqrt{1+t^4}\,dt$

$u = 1+t^4$
$du = 4t^3\,dt$

$g'(x) = (x^2)^3\sqrt{1+(x^2)^4} \cdot \frac{d}{dx}(x^2)$

$= x^6\sqrt{1+x^8} \cdot 2x$

$\boxed{= 2x^7\sqrt{1+x^8}}$

$= \frac{1}{4}\int_2^{1+a^4} u^{1/2}\,du$

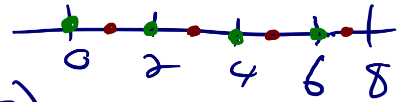
$= \frac{1}{4} \cdot \frac{2}{3} u^{3/2} \Big|_2^{1+a^4}$

$\boxed{= \frac{1}{6}\left((1+a^4)^{3/2} - 2^{3/2}\right)}$

Example 4: Estimate the area under the curve $y = x^2 + 2$ on the interval $[0, 8]$ using 4 sub-intervals and the method given below.

a) left endpoints.

b) midpoints.



$L_4 = 2(f(0) + f(2) + f(4) + f(6))$

$= 2(2 + 6 + 18 + 38)$

$= \boxed{128}$

$M_4 = 2(f(1) + f(3) + f(5) + f(7))$

$\Delta x = \frac{8-0}{4} = 2$

$= 2(3 + 11 + 27 + 49)$

$= \boxed{180}$

Example 5: Evaluate the following definite integrals.

a) $\displaystyle\int_0^{\pi/4} \frac{\sec^2 t}{\tan t + 1} dt$

$u = \tan t + 1$

$du = \sec^2 t$

$= \int_1^2 \frac{1}{u} du$

$= \ln|u| \Big|_1^2$

$= \ln 2 - \ln 1$

$= \boxed{\ln 2}$

b) $\displaystyle\int_1^4 \frac{x-2}{\sqrt{x}} dx$ $= \int_1^4 (x^{1/2} - 2x^{-1/2}) dx$

$= \frac{2}{3}x^{3/2} - x^{1/2} \Big|_1^4$

$= \boxed{2/3}$

Example 6: Evaluate the following indefinite integrals.

a) $\displaystyle\int \frac{\sin(1/x)}{x^2} dx$

$u = 1/x$

$du = -\frac{1}{x^2} dx$

$= -\int \sin u \, du$

$= \cos u + C$

$= \boxed{\cos(1/x) + C}$

b) $\displaystyle\int \frac{x}{(x-2)^3} dx$

$u = x - 2$

$x = u + 2$

$du = dx$

$= \int \frac{u+2}{u^3} du$

$= \int \left(\frac{1}{u^2} + \frac{2}{u^3}\right) du$

$= -\frac{1}{u} - \frac{1}{u^2} + C$

$= \boxed{-\frac{1}{x-2} - \frac{1}{(x-2)^2} + C}$

Example 7: A particle moves along a line with velocity function $v(t) = 2\sin t$, where v is measured in meters per second.

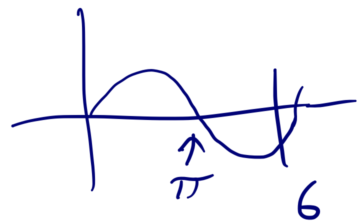
(a) Find the displacement over the time interval $[0, 6]$

$$s = \int_0^6 2\sin t \, dt$$

$$= -2\cos t \Big|_0^6 = -2(\cos 6 - \cos 0)$$

$$= \boxed{-2\cos 6 + 2 \text{ m}}$$

(b) Find the total distance traveled during the time interval $[0, 6]$



$$d = \int_0^6 |2\sin t| \, dt$$

$$= \int_0^{\pi} 2\sin t \, dt + \int_{\pi}^6 -2\sin t \, dt$$

$$= -2\cos t \Big|_0^{\pi} + 2\cos t \Big|_{\pi}^6$$

$$= -2(\cos \pi - \cos 0) + 2(\cos 6 - \cos \pi)$$

$$= 4 + 2\cos 6 + 2 = \boxed{6 + 2\cos 6 \text{ m}}$$

Example 8: A bacteria population is 4000 at time $t = 0$ and its rate of growth is 1000×2^t bacteria per hour after t hours. What is the population after one hour?

$$\text{pop} = 4000 + \int_0^1 1000 \cdot 2^t \, dt$$

$$= 4000 + \frac{1000 \cdot 2^t}{\ln 2} \Big|_0^1$$

$$= 4000 + \frac{1000}{\ln 2}(2^1 - 2^0)$$

$$= 4000 + \frac{1000}{\ln 2} \approx \boxed{5{,}443 \text{ bacteria}}$$