Your Name

Solutions ✓

Your Signature

Instructor Name

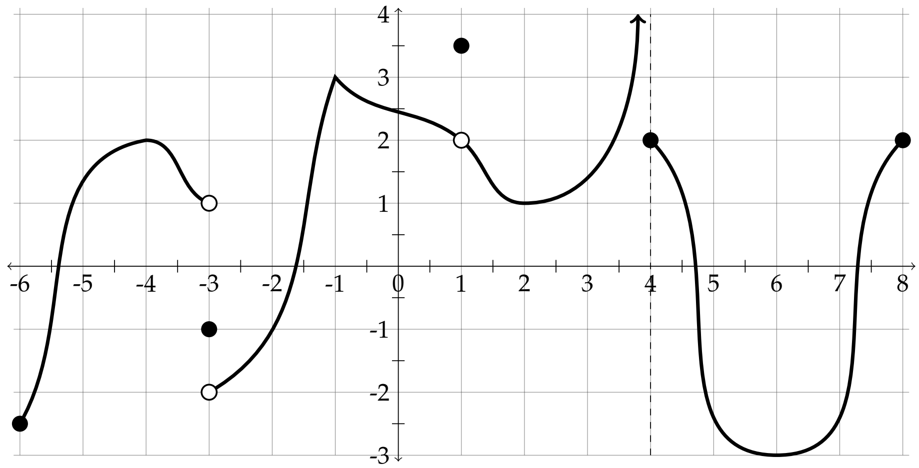
End Time

Problem	Total Points	Score
1	6	
2	15	
3	6	
4	7	
5	12	
6	6	
7	6	
8	16	
9	16	
10	5	
11	5	
Extra Credit	(6)	
Total	100	

- The total time allowed for this exam is two hours.
- This test is closed notes and closed book.
- You may **not** use a calculator.
- In order to receive full credit, you must **show your work.** Be wary of doing computations in your head. Instead, write out your computations on the exam paper.
- **PLACE A BOX AROUND** **YOUR FINAL ANSWER** **to each question** where appropriate.
- If you need more room, use the backs of the pages and indicate to the reader that you have done so.
- Raise your hand if you have a question.

1 (6 points) The graph of the function $f(x)$ given below has domain $[-6, 8]$. Use it to answer the questions below.

If you are asked to determine a limit, find the limit or one-sided limit as directed. Use ∞ and $-\infty$ where appropriate. If the limit does not exist and cannot be described using ∞ or $-\infty$, write "DNE".



2

(a) $\lim_{x\to-3^-} f(x) = \boxed{1}$

(b) $\lim_{x\to1} f(x) = \boxed{2}$

(c) $\lim_{x\to-3} f(x) = \boxed{\text{DNE}}$

(d) $\lim_{x\to4^-} f(x) = \boxed{\infty}$

(e) At what x-values in its domain is $f(x)$ NOT continuous? If f is continuous everywhere on its domain, write "none".

1 $\boxed{\text{at } x = -3, 1, 4}$

(f) At what x-values in its domain is $f(x)$ NOT differentiable? If f is differentiable everywhere on its domain, write "none".

1 $\boxed{\text{at } x = -3, -1, 1, 4,}$

(g) What are the x-values corresponding to local maxima of $f(x)$? If there aren't any, write "none".

1 $\boxed{\text{at } x = -4, -1, 1, 8 \leftarrow \text{optional}}$

(h) What are the x-values corresponding to absolute maxima of $f(x)$? If there aren't any, write "none".

1 $\boxed{\text{none}}$ as $f(x) \to \infty$ as $x \to 4^-$

2 (15 points) Evaluate the following limits. Show all work and explain your reasoning algebraically or in words, when applicable.

(a) $\lim_{x\to-\infty} \frac{\sqrt{9x^6+1}}{x^3+5} = \lim_{x\to\infty} \frac{\sqrt{9(-x)^6+1}}{(-x)^3+5}$

$= \lim_{x\to\infty} \frac{(\sqrt{9x^6+1})\,1/x^3}{(-x^3+5)\,1/x^3}$

$= \lim_{x\to\infty} \frac{\sqrt{9+1/x^6}}{-1+5/x^3}$

$= \frac{\sqrt{9}}{-1}$

$= \boxed{-3}$

① adress $-\infty$
② correct selection/ algebra w/ $1/x^3$
① ans.

(b) $\lim_{x\to 0} \frac{\arcsin(4x)}{x} \overset{H}{=} \lim_{x\to 0} \left(\frac{\frac{4}{\sqrt{1-(4x)^2}}}{1}\right)$

"$\frac{0}{0}$ form"

$= \lim_{x\to 0} \frac{4}{\sqrt{1-16x^2}}$

$= \frac{4}{\sqrt{1}}$

$= \boxed{4}$

① identify L'H
② apply L'H correctly
① answer

(c) $\lim_{x\to 9^-} \frac{\sqrt{x}}{(x-9)^3} = \boxed{-\infty}$

as $x \to 9^-$, $\sqrt{x} \to 3$

as $x \to 9^-$, $x-9 \to$ small negative and $(x-9)^3$ is also a small negative.

① numerator $\to 3$
① denom $\to 0^-$
① ans $= -\infty$

(d) $\lim_{x\to 0^+} (1-2x)^{1/x}$

$y = (1-2x)^{1/x}$

$\ln y = \ln(1-2x)^{1/x}$

$\ln y = \frac{\ln(1-2x)}{x}$ ① log

$\lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{\ln(1-2x)}{x}$

$= \lim_{x\to 0} \left(\frac{\frac{-2}{1-2x}}{1}\right)$ ① correct L'H

$= -2$

① ans

Since $\lim_{x\to 0} \ln y = -2$, then

$\lim_{x\to 0} y = \boxed{e^{-2} = 1/e^2}$ ① ans

3 (6 points)

(a) Complete the definition of the derivative of a function $f(x)$ below:

2

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

(b) Find the derivative of $f(x) = 5x^2 - x$ using the definition of the derivative. You must show your work to receive credit.

$$f'(x) = \lim_{h\to 0} \frac{5(x+h)^2 - (x+h) - (5x^2 - x)}{h}$$

1 input

$$= \lim_{h\to 0} \frac{5(x^2+2xh+h^2) - x - h - 5x^2 + x}{h}$$

$$= \lim_{h\to 0} \frac{5x^2 + 10xh + 5h^2 - x - h - 5x^2 + x}{h}$$

+1 algebra in middle

$$= \lim_{h\to 0} \frac{10xh + 5h^2 - h}{h}$$

$$= \lim_{h\to 0} (10x + 5h - 1)$$ +1 full simplify

$$= \boxed{10x - 1}$$

+1 ans

4 (7 points) The volume of a circular cylinder is increasing at a rate of 20π m^3/sec while the radius is increasing at a rate of 2 m/sec. How must the height of the cylinder be changing when the volume is 90π m^3 and the radius is 3 m? Include units with your answer. ($V = \pi r^2 h$)

Know: $\frac{dV}{dt} = 20\pi$ m^3/sec

$\frac{dr}{dt} = 2$ m/sec.

(1) identify & input correctly

Want: $\frac{dh}{dt}$ when $V = 90\pi$, $r = 3$,

note $90\pi = \pi \cdot 3^2 \cdot h \Rightarrow 90\pi = 9\pi h$

$\Rightarrow h = 10$ (1) find h

$$V = \pi r^2 h$$

$$\frac{dV}{dt} = 2\pi r \frac{dr}{dt} h + \pi r^2 \frac{dh}{dt}$$

(3) for correct diff
−2 no prod rule

$$20\pi = 2\pi(3)(2)(10) + \pi \cdot 3^2 \cdot \frac{dh}{dt}$$

$$20\pi = 120\pi + 9\pi \frac{dh}{dt}$$

$$-100\pi = 9\pi \frac{dh}{dt}$$

$$\boxed{\frac{dh}{dt} = \frac{-100}{9} \text{ m/sec}}$$

(1) ans
(1) units

The height is decreasing at a rate of 100/9 meters per second.

5 (12 points) Calculate the derivatives of the given functions. Do not simplify your answers, but use parentheses appropriately.

(a) $y = (x^2+1)^{\cos x}$

$$\ln y = \ln (x^2+1)^{\cos x}$$

(+1) for log

$$\ln y = \cos x \cdot \ln(x^2+1)$$

(+2) for deriv

$$\frac{1}{y} y' = -\sin x \cdot \ln(x^2+1) + \cos x \cdot \frac{2x}{x^2+1}$$

$$\boxed{y' = \left(\frac{2x \cos x}{x^2+1} - \sin x \ln(x^2+1)\right) (x^2+1)^{\cos x}}$$

(+1) ans

(b) $g(z) = \dfrac{\sec(8z)}{1+z^2}$

$$\boxed{g'(z) = \frac{(1+z^2) \cdot 8 \cdot \sec(8z)\tan(8z) - 2z\sec(8z)}{(1+z^2)^2}}$$

$$g'(z) = \frac{2\sec(8z)\left(4(z^2+1)\tan(8z) - z\right)}{(1+z^2)^2}$$

if they do take out GCF.

(+1) OR
(+1) deriv of sec
(+1) chain w/ sec
(+1) answer

(c) $h(x) = \displaystyle\int_{\arctan x}^{5} \sqrt{3+2t^3}\,dt = -\int_5^{\arctan x} \sqrt{3+2t^3}\,dt$

$$h'(x) = \frac{d}{dx}\left(-\int_5^{\arctan x} \sqrt{3+2t^3}\,dt\right)$$

(+1) neg
(+2) input
(+1) chain rule

$$= \boxed{-\sqrt{3+2(\arctan x)^3} \cdot \left(\frac{1}{1+x^2}\right)}$$

$$= \boxed{-\frac{\sqrt{3+2(\arctan x)^3}}{1+x^2}}$$

6 (6 points) Let $f(x) = e^{4x}\cos x$.

(a) (4 points) Find the linearization of the function $f(x)$ at the point $a = 0$.

$f'(x) = 4e^{4x}\cos x - e^{4x}\sin x$ (+1)

Point $a=0$, $f(0) = e^0\cos 0 = 1$ (+1)

Slope $m = f'(0) = 4e^0\cos 0 - e^0\sin 0 = 4$ (+1)

equation: $y - y_1 = m(x - x_1)$

$\boxed{y - 1 = 4(x - 0)}$

$\boxed{y = 4x + 1}$ ✓ (+1)

(b) (2 points) Use your linear approximation from part (a) to estimate $f(0.1)$.

$f(0.1) \approx 4(0.1) + 1$

$= \boxed{1.4}$ ✓

7 (6 points) Find the absolute maximum and absolute minimum of $f(x) = x^3 - 3x + 5$ on the interval $[0, 3]$.

$f(x) = x^3 - 3x + 5$

$f'(x) = 3x^2 - 3$ (+1)

$0 = 3(x^2 - 1)$

$0 = 3(x-1)(x+1)$

$x = 1$, $x = -1$

not in the domain

(+1) for CN

$f(0) = 5$ (+1)

$f(1) = 1 - 3 + 5 = 3$ (+1)

$f(3) = 27 - 9 + 5 = 23$ (+1)

$\boxed{\text{abs min } f(1) = 3}$

$\boxed{\text{abs max } f(3) = 23}$ ✓

[-1 for inputting $x=-1$]

ans (+1)

8 (16 points) Answer the following questions using the given function and its derivatives. **Note that this problem continues onto the next page.**

$$f(x) = \frac{3x^2 - 1}{x^3}, \qquad f'(x) = \frac{-3(x^2-1)}{x^4}, \qquad f''(x) = \frac{6(x^2-2)}{x^5}$$

(a) Find the vertical asymptotes, if any.

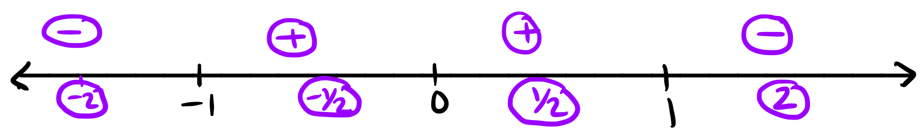
1 $\boxed{x = 0}$

(b) Find the horizontal asymptotes, if any.

1 $\boxed{y = 0}$

(c) Find the intervals of increase or decrease.

$f'(x) = 0$ / undefined at $x = \pm 1, 0$



sign f'

$\boxed{\begin{array}{l} \text{inc on } (-1,0) \cup (0,1) \\ \text{dec on } (-\infty,-1) \cup (1,\infty) \end{array}}$

1 find CN
1 sign analysis
1 answer

(d) Find and classify the local maximum and minimum values, if any.

$f(-1) = \frac{3-1}{(-1)} = \boxed{-2 \text{ is a local min}}$

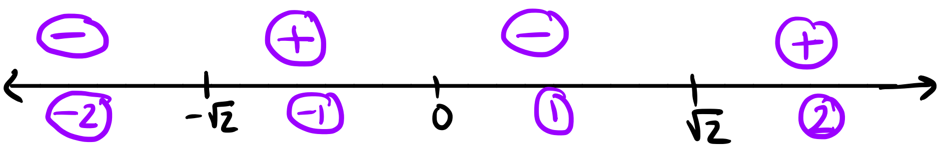
$f(1) = \frac{3-1}{1} = \boxed{2 \text{ is a local max}}$

2 for y's
1 for classifying

(e) Find the intervals of concavity and the x-**values only** of the inflection points.

$f''(x) = 0$ / undef @ $x = 0$ and $\pm\sqrt{2}$

2 for CN's and sign analysis



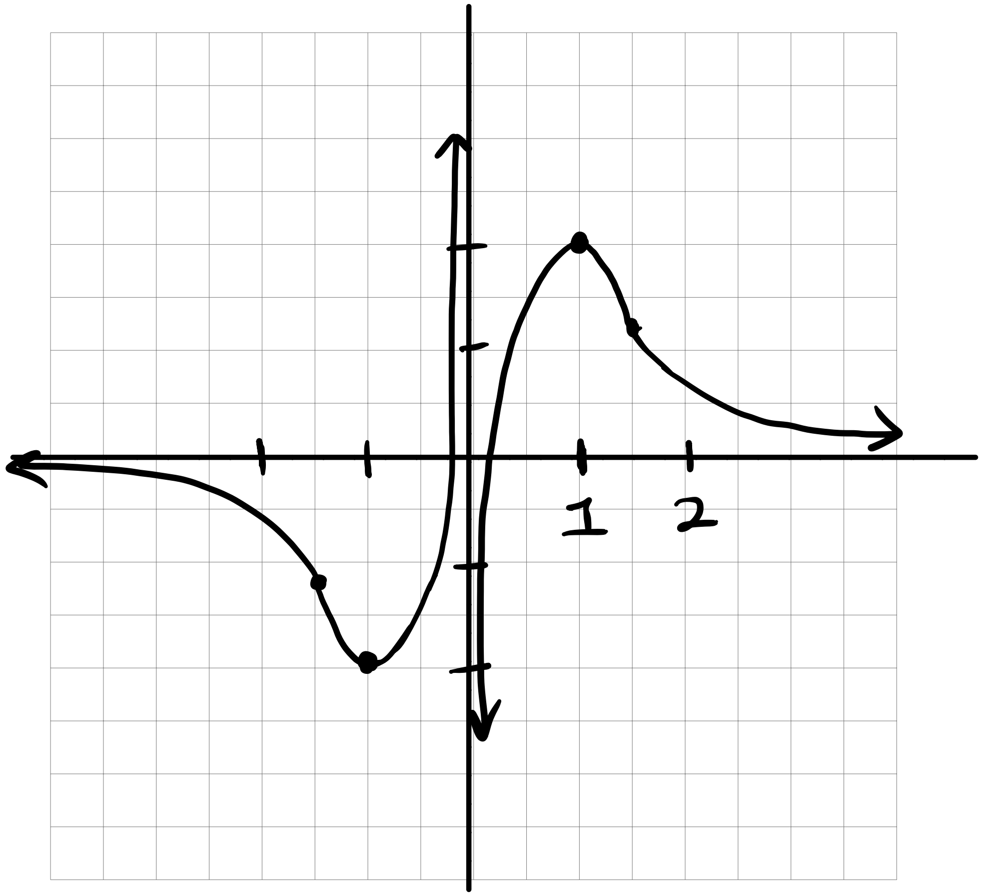
$\boxed{\begin{array}{l} \text{CU on } (-\sqrt{2},0) \cup (\sqrt{2},\infty) \\ \text{CD on } (-\infty,-\sqrt{2}) \cup (0,\sqrt{2}) \end{array}}$

$\boxed{\text{IP at } x = \sqrt{2}, -\sqrt{2}}$

1 for IPs

1 for intervals

(f) Use the information from parts (a) - (e) to sketch the graph. $\sqrt{2} \approx 1.41$



Zen +4 right
+3 very close
+2 a few correct elements
+1 something right
+0 whaa??

−1 miss 2 or more +C

9 (16 points) Evaluate the following integrals.

(a) $\int \frac{1+x^2}{x^{5/3}}dx = \int (x^{-5/3} + x^{+1/3})\,dx$

$= \boxed{-\frac{3}{2}x^{-2/3} + \frac{3}{4}x^{4/3} + C}$

$= \boxed{-\frac{3}{2x^{2/3}} + \frac{3x^{4/3}}{4} + C}$

+1 algebra

+3 correct powers/coefficients

(b) $\int \frac{x\sin(x^2)}{8}dx = \int \frac{x \sin u}{8} \cdot \frac{du}{2x}$

$u = x^2$
$du = 2x\,dx$

$= \frac{1}{16}\int \sin u \, du$

$= -\frac{1}{16}\cos u + C$

$= \boxed{-\frac{1}{16}\cos(x^2) + C}$

+1 for sub
+1 for 1/16
+1 for $-\cos(u)$
+1 backsubstitute

(c) $\int_e^{e^4} \frac{4\,dx}{x(\ln x)^3} = \int_1^4 \frac{4}{u^3}du = \boxed{15/8}$

$u = \ln x$
$du = 1/x \, dx$
$x = e, \; u = 1$
$x = e^4, \; u = 4$

$= \left.\frac{4u^{-2}}{-2}\right|_1^4$

$= -2(1/16 - 1)$

$= -2(-15/16)$

+1 for sub
+1 deal w/bounds
+1 antideriv
+1 ans.

(d) $\int 2x\sqrt{x+5}\,dx = 2\int (u-5)\sqrt{u}\,du$

$u = x+5$
$du = dx$
$x = u-5$

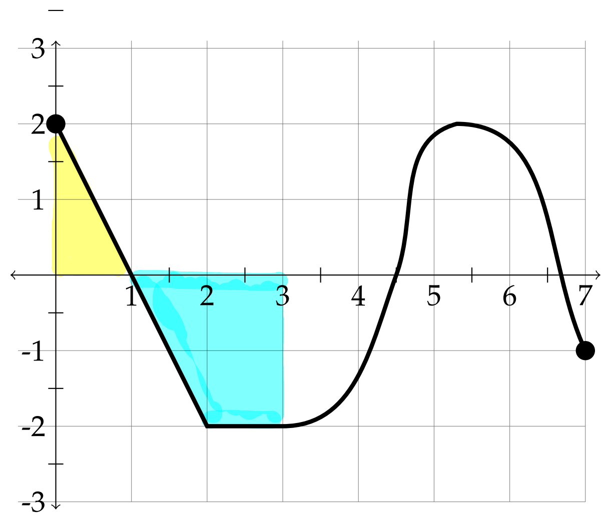
$= 2\int (u^{3/2} - 5u^{1/2})\,du$

$= 2\left(\frac{2}{5}u^{5/2} - 5 \cdot \frac{2}{3}u^{3/2}\right) + C$

$= \boxed{\frac{4}{5}(x+5)^{5/2} - \frac{20}{3}(x+5)^{3/2} + C}$

+1 sub
+1 clever subbing
+1 antideriv
+1 back sub

$\boxed{10}$ (5 points) The graph of $f(x)$ is given below.



(a) Evaluate $\int_0^1 f(x)\,dx$. $= \frac{1}{2}\cdot 1\cdot 2$

$= \boxed{1}$ ✓ (+1)

(b) Evaluate $\int_0^3 f(x)\,dx$. $= \frac{1}{2}(1)(2) - \frac{1}{2}(1)(2) - 2(1)$

$= \boxed{-2}$ ✓ (+1)

(c) Where does $g(x) = \int_0^x f(t)\,dt$ achieve a local maximum on the interval $(0, 7)$? Justify your answer.

when $g'(x) = f(x) = 0$ and changes from ⊕ to ⊖ (+1) justify

true for $\boxed{x = 1 \text{ and } x \approx 6.7}$ ✓ (+1)

(d) Are there any values of x such that $g(x) = \int_0^x f(t)\,dt = 0$ on $[0, 7]$?

$\boxed{\text{at } x = 2}$ ✓ (+1)

11 (5 points) Water flows into a reservoir at a rate of $1000 - 20t$ liters per hour.

(a) What does the quantity $\int_1^5 (1000 - 20t)\, dt$ represent?

2

The amount of water that flowed into the reservoir between $t=1$ hours and $t=5$ hours.

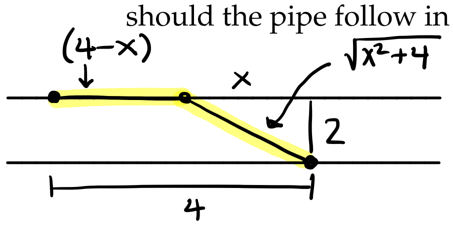
(b) Assume the reservoir initially contained 50,000 liters, how much water is in the reservoir after 2 hours?

$$\int_0^2 (1000-20t)\,dt = (1000t - 10t^2)\Big|_0^2$$
$$= 2000 - 40$$
$$= 1960 \text{ liters added} \quad +2$$

for a total of $50000 + 1960 = \boxed{51{,}960 \text{ liters}}$ +1

12 (6 points) **[Extra Credit]** An offshore oil well is 2 kilometers off the coast. The refinery is 4 kilometers down the coast. Laying pipe in the ocean is twice as expensive as on land. What path should the pipe follow in order to minimize the cost?

+1



$C = (4-x) + 2\sqrt{x^2+4}$ +1

$C' = -1 + 2\cdot\frac{1}{2}(x^2+4)^{-1/2}\cdot 2x$

$0 = -1 + \dfrac{2x}{\sqrt{x^2+4}}$ +1

$1 = \dfrac{2x}{\sqrt{x^2+4}}$

$\sqrt{x^2+4} = 2x$

$x^2 + 4 = 4x^2$

$4 = 3x^2$

$4/3 = x^2$

$x = \pm 2/\sqrt{3}$ +1

$\boxed{x = 2/\sqrt{3}}$ +1

sign C'

$\ominus$ $\oplus$

0 $2/\sqrt{3}$ 4

By the first derivative test we have a min at $x = 2/\sqrt{3}$. +1