Your Name

Solutions

Your Signature

Instructor Name

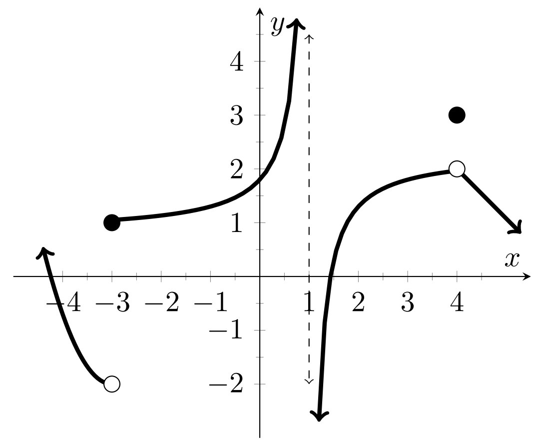
End Time

Problem	Total Points	Score
1	8	
2	10	
3	18	
4	18	
5	10	
6	8	
7	8	
8	6	
9	10	
10	4	
Extra Credit	(5)	
Total	100	

- This test is closed notes and closed book.
- You may **not** use a calculator.
- In order to receive full credit, you must **show your work.** Be wary of doing computations in your head. Instead, write out your computations on the exam paper.
- **PLACE A BOX AROUND** **YOUR FINAL ANSWER** **to each question** where appropriate.
- If you need more room, use the backs of the pages and indicate to the reader that you have done so.
- Raise your hand if you have a question.

1 (8 points)

For the function $f(x)$ whose graph is given below, state the value of each quantity if it exists.



(a) $\lim_{x\to -3^-} f(x) =$ -2

(b) $\lim_{x\to -3^+} f(x) =$ 1

(c) $\lim_{x\to -3} f(x) =$ DNE

(d) $f(-3) =$ 1

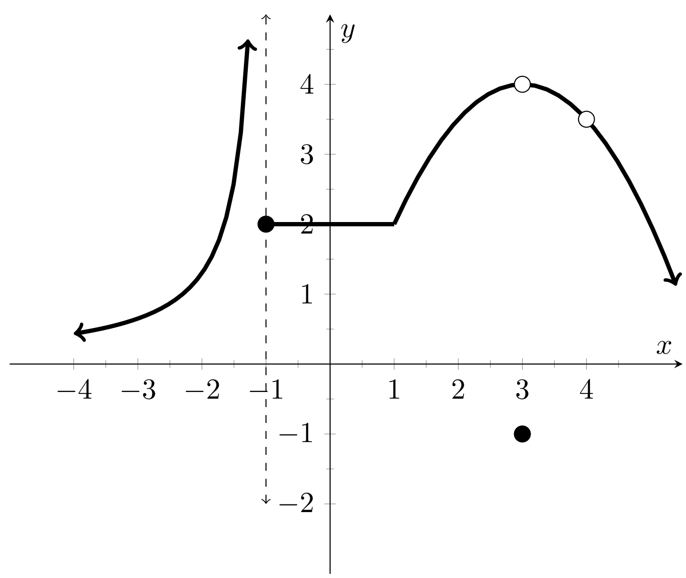
(e) $\lim_{x\to 1^-} f(x) =$ ∞

(f) $\lim_{x\to 1^+} f(x) =$ $-\infty$

(g) $\lim_{x\to 4} f(x) =$ 2

(h) $f(4) =$ 3

2 (10 points) A graph of the function $f(x)$ is displayed below.



(a) (6 points) From the graph of f, state the numbers at which f is discontinuous and why.

$x=-1$: $\lim_{x\to -1^-} f(x) = \infty$

$x=4$: $f(4)$ DNE

$x=3$: $\lim_{x\to 3} f(x) = 4 \neq -1 = f(3)$

(b) (4 points) From the graph of f, state the numbers at which f fails to be differentiable and why.

$x=-1,3,4$: f is not continuous

$x=1$: f has a corner

3 (18 points) Evaluate the following limits. Justify your answers with words and/ or any relevent algebra. Be sure to use proper notation, as points will be deducted for not doing so.

(a) $\lim_{x\to-2} \frac{x^2+2x}{x^2-3x-10} = \lim_{x\to-2} \frac{x(x+2)}{(x-5)(x+2)} = \lim_{x\to-2} \frac{x}{x-5}$

$= \frac{-2}{-2-5} = \boxed{\frac{2}{7}}$

(b) $\underbrace{\lim_{x\to1} \ln\left(\frac{7-x^2}{1+x}\right)} = \underbrace{\ln\left(\frac{7-1^2}{1+1}\right)} = \ln\left(\frac{6}{2}\right) = \boxed{\ln 3}$

I observe that all the pieces of this function are continuous when x=1. So I am allowed to do this.

(c) $\lim_{x\to\infty} \frac{1+x-2x^2}{x^3+1} \cdot \frac{1/x^3}{1/x^3} = \lim_{x\to\infty} \frac{\frac{1}{x^3}+\frac{1}{x^2}-\frac{2}{x}}{1+1/x^3} = \frac{0+0-0}{1+0} = \frac{0}{1} = \boxed{0}$

4 (18 points) Evaluate the following limits. Justify your answers with words and/ or any relevent algebra. Be sure to use proper notation, as points will be deducted for not doing so.

(a) $\lim_{x\to 4^+} \frac{-\sqrt{x}}{(4-x)^3} = \infty$

As x approaches x from above, $-\sqrt{x}$ approaches -2 and $(4-x)^3$ approaches 0 but is always negative. So the quotient is unbounded and positive.

(b) $\lim_{x\to 4} \frac{\frac{1}{x^2}-\frac{1}{16}}{x-4} = \lim_{x\to 4} \left(\frac{1}{x-4}\right)\left(\frac{16-x^2}{x^2\cdot 16}\right)$

$= \lim_{x\to 4} \left(\frac{1}{x-4}\right)\left(\frac{(4-x)(4+x)}{16x^2}\right)$ $4-x = -(x-4)$.

$= \lim_{x\to 4} \frac{-(x-4)(x+4)}{(x-4)(16x^2)}$

$= \lim_{x\to 4} \frac{-(x+4)}{16x^2} = \frac{-8}{16\cdot 16} = \boxed{\frac{-1}{32}}$

(c) $\lim_{x\to -\infty} \frac{\sqrt{7+25x^6}}{2+x^3} = \lim_{x\to\infty} \frac{\sqrt{7+25x^6}}{2-x^3} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}}$

$= \lim_{x\to\infty} \frac{\sqrt{\frac{7}{x^6}+25}}{\frac{2}{x^3}-1} = \frac{\sqrt{0+25}}{0-1} = \boxed{-5}$

$\frac{1}{x^3} = \sqrt{\frac{1}{x^6}}$ for $x \geq 0$.

5 (10 points)

Given $f(x) = \begin{cases} 3 & x \geq 3 \\ \frac{3x-9}{|x-3|} & x < 3 \end{cases}$ find $\lim_{x\to 3} f(x)$ or explain why this limit does not exist.

answer:

$\lim_{x\to 3} f(x)$ does not exist because

$$\lim_{x\to 3^+} 3 = 3 \neq \boxed{-3 = \lim_{x\to 3^-} f(x).}$$

* How do I know this?

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{3x-9}{|x-3|} = \lim_{x\to 3^-} \frac{3(x-3)}{-(x-3)} = \lim_{x\to 3^-} -3 = -3$$

If x is less than 3, then $x-3<0$. So $|x-3| = -(x-3)$.

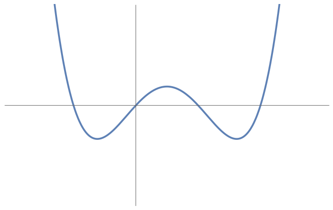
6 (8 points)

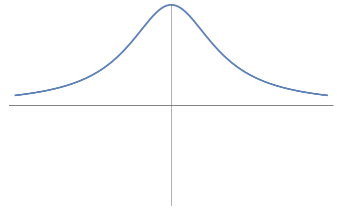
Using complete sentences, use the Interemdiate Value Theorem to show that there is a root of the equation $2e^x = 1 + 8x$ in the interval $(0, 1)$.

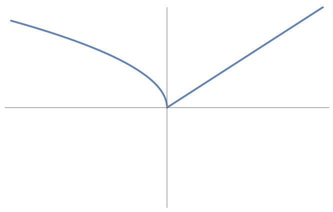
Let $f(x) = 2e^x - 8x - 1$ which is continuous. Observe that $f(0) = 2e^0 - 0 - 1 = 2 - 1 = 1 > 0$ and $f(1) = 2e - 8 - 1 = 2e - 9 < 0$. So the Intermediate Value Theorem tells us that there is some c-value in $(0,1)$ where $f(c) = 0$. So c is a solution to the equation.

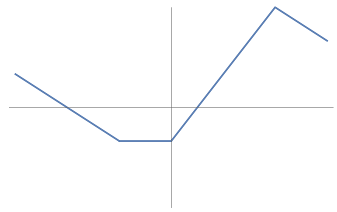
* crucial parts of a correct answer.

$\boxed{7}$ (8 points) Match the graph of each function (a) - (d) with the graph of its derivative I-VIII. Please put your answers in the blanks provided below the graphs.


a.)


c.)


b.)

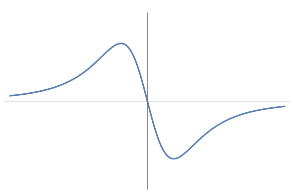

d.)

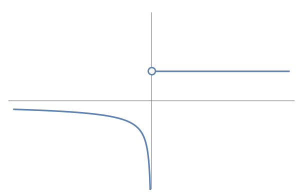
(a) Graph (a)'s derivative is given by VI

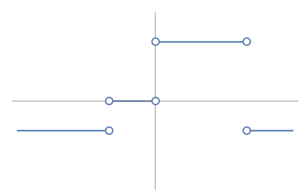
(b) Graph (b)'s derivative is given by II

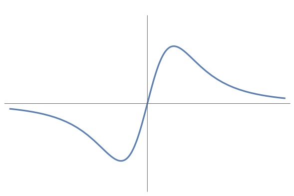
(c) Graph (c)'s derivative is given by I

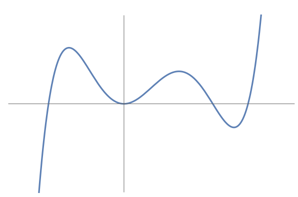
(d) Graph (d)'s derivative is given by III

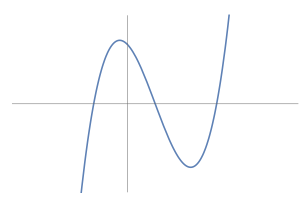

I.

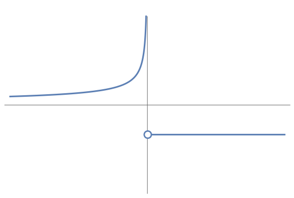

II.

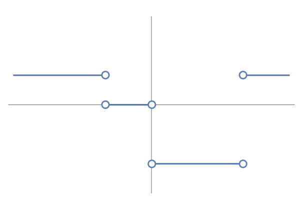

III.


IV.


V.


VI.


VII.


VIII.

8 (6 points)

Given $f(x) = \frac{8}{x^2}$ the derivative of $f(x)$ is given by $f'(x) = -\frac{16}{x^3}$. Using this derivative find the equation of the tangent line to $f(x)$ when $x = 2$. Give your final answer in slope-intercept form.

$f(2) = \frac{8}{4} = 2$, $f'(2) = \frac{-16}{8} = -2$; point $(2,2)$; slope $m = -2$.

$y - 2 = -2(x-2)$

answer: $\boxed{y = 2x + 6}$

9 (10 points)

(a) (2 points) State the limit definition of the derivative of the function $f(x)$.

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

(b) (8 points)
Given $f(x) = \sqrt{7x}$, find $f'(x)$ using the definition. **No credit will be given for answers found using derivative short-cut formulas**

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{7(x+h)} - \sqrt{7x}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{7x+7h} - \sqrt{7x}}{h} \cdot \frac{(\sqrt{7x+7h} + \sqrt{7x})}{(\sqrt{7x+7h} + \sqrt{7x})}$$

$$= \lim_{h\to 0} \frac{7x+7h-7x}{h(\sqrt{7x+7h} + \sqrt{7x})}$$

$$= \lim_{h\to 0} \frac{7h}{h(\sqrt{7x+7h} + \sqrt{7x})}$$

$$= \lim_{h\to 0} \frac{7}{\sqrt{7x+7h} + \sqrt{7x}} = \frac{7}{\sqrt{7x} + \sqrt{7x}} = \boxed{\frac{7}{2\sqrt{7x}}}$$

$\boxed{10}$ (4 points) The number of bacteria after t hours in a controlled laboratory setting is given by the function $n = f(t)$ where n is the number of bacteria and t is measured in hours.

(a) Suppose $f'(10) = -300$. What are the units of the derivative?

$$\text{units } f'(x) = \frac{\text{units } f}{\text{units } x} = \frac{\text{bacteria}}{\text{hours}}$$

(b) In the context of this situation, explain what $f'(10) = -300$ means using complete sentences.

After 10 hours, the population of bacteria is decreasing at a rate of 300 bacteria per hour.

$\boxed{11}$ (5 points)

Prove that $\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$. You must clearly explain your work and cite any relevent theorems for full credit.

Let $f(x) = x^2$ and $g(x) = -x^2$.

Then for all x^2, $-x^2 \le x^2 \sin\left(\frac{1}{x}\right) \le x^2$.

So, applying the Squeeze Theorem, we find

$$\lim_{x\to 0} -x^2 \le \lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) \le \lim_{x\to 0} x^2.$$

Thus $$0 \le \lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) \le 0.$$

So $$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0.$$