Your Name

Solutions

Your Signature

Instructor Name

End Time

| Problem | Total Points | Score |
|---|---|---|
| 1 | 8 | |
| 2 | 8 | |
| 3 | 8 | |
| 4 | 8 | |
| 5 | 16 | |
| 6 | 10 | |
| 7 | 10 | |
| 8 | 15 | |
| 9 | 10 | |
| 10 | 7 | |
| Total | 100 | |

- The total time allowed for this exam is 60 minutes.
- This test is closed notes and closed book.
- You may **not** use a calculator.
- In order to receive full credit, you must **show your work.** Be wary of doing computations in your head. Instead, write out your computations on the exam paper.
- **PLACE A BOX AROUND** $\boxed{\textbf{YOUR FINAL ANSWER}}$ **to each question** where appropriate.
- If you need more room, use the backs of the pages and indicate to the reader that you have done so.
- Raise your hand if you have a question.

product rule

1 (8 points) Find $dy/dx$ when $\boxed{3xy} + 2x^2 - y^2 = 1$. ← implicit differentiation!

$$3 \cdot y + 3x \cdot y' + 4x - 2yy' = 0$$

$$3xy' - 2yy' = -4x - 3y$$

$$(3x - 2y)y' = -4x - 3y$$

$$\boxed{y' = \frac{-4x-3y}{3x-2y} = \frac{4x+3y}{2y-3x}}$$

either answer ok

2 (8 points) Given $y = x^{\cos x}$ find $y'$. ← logarithmic differentiation

$$y = x^{\cos x}$$

$$\ln y = \cos x \ln x$$ ← product rule

implicit differentiation again!

$$\frac{1}{y} y' = -\sin x \cdot \ln x + \cos x \cdot \frac{1}{x}$$

$$y' = y\left((-\sin x)(\ln x) + \frac{\cos x}{x}\right)$$

$$\boxed{y' = \left(x^{\cos x}\right)\left[\frac{\cos x}{x} - (\sin x)\ln x\right]}$$

replace "y" with its expression using x's.

3 (8 points)

math-eze for "Find the tangent line."

(a) Find the linearization of $f(x) = \sqrt{7+x^2}$ at $a = 3$.

$f'(x) = \frac{1}{2}(7+x^2)^{-1/2}(2x)$

$= \frac{x}{\sqrt{7+x^2}}$

$m = f'(3) = \frac{3}{\sqrt{7+9}} = \frac{3}{4}$

$f(3) = \sqrt{7+9} = 4$

point $(3,4)$

tangent line: $y - 4 = \frac{3}{4}(x-3)$

Answer:

$y = \frac{3}{4}(x-3) + 4$ or

$L(x) = \frac{3}{4}(x-3) + 4.$

(b) Use linear approximation to estimate the value of $f(x)$ at $a = 3.1$.

Use the tangent line to estimate the function.

$f(3.1) \approx L(3.1) = \frac{3}{4}(3.1-3) + 4 = (0.75)(0.1) + 4$

$= \boxed{4.075}$

4 (8 points) Find the absolute maximum and minimum of the function $f(x) = \frac{1}{3}x^3 + 2x^2 - 12x + 1$ on the interval $0 \le x \le 3$.

closed interval method. Yay!

Find critical numbers:

$f'(x) = x^2 + 4x - 12 = (x+6)(x-2) = 0$

$\cancel{x = -6}$ or $x = 2$

not in domain

Make chart:

| $x$ | $f(x)$ |
|---|---|
| 0 | $f(0) = 1$ ← largest |
| 2 | $f(2) = 8/3 + 8 - 24 + 1 = -12\,1/3$ ← smallest. |
| 3 | $f(3) = 9 + 18 - 36 + 1 = -8$ |

$$\begin{array}{r} -24 \\ +11\,2/3 \\ \hline -13 + 2/3 \end{array}$$

$$\begin{array}{r} -36 \\ +28 \\ \hline -8 \end{array}$$

Answer:

absolute max is 1

absolute min is $-12\,1/3$

5 (16 points) Evaluate the following limits.

(a) $\lim_{x\to 0} \frac{1-\cos x}{x^2} \overset{H}{=} \lim_{x\to 0} \frac{\sin x}{2x} = \lim_{x\to 0} \frac{-\cos x}{2} = \frac{-1}{2}$

form $\frac{0}{0}$ form $\frac{0}{0}$ just plug in.

(b) $\lim_{t\to 0} \frac{t^2+3}{\cos t} = \frac{0+3}{\cos 0} = \frac{3}{1} = 3.$ (No L'Hospital's Rule needed here ☺)

form $\infty^0$

(c) $\lim_{x\to\infty} (x^2)^{1/x} = \boxed{e^0} = 1$

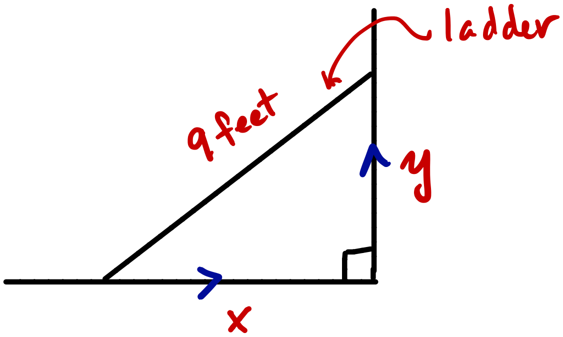
$y = (x^2)^{1/x} = x^{2/x}$; $\ln y = \frac{2}{x} \ln x$.

$\lim_{x\to\infty} \frac{2\ln x}{x} \overset{H}{=} \lim_{x\to\infty} \frac{2\cdot\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{2}{x} = 0$

form $\frac{\infty}{\infty}$

$\boxed{6}$ (10 points) A 9 foot ladder is resting against the wall. The bottom is initially 8 feet away from the wall and is being pushed towards the wall at a rate of 1 ft/sec.

(a) Sketch and label a diagram modeling the situation described above.




$\frac{dx}{dt} = -1$ ft/s

x starts at $x=8$

(b) How fast is the top of the ladder moving up the wall 4 seconds after we start pushing? Give your answer using appropriate units.

We want $\frac{dy}{dt}$ when $t=4$

$\boxed{7}$ (10 points) Sketch the graph of a function $f(x)$ that satisfies all of the given conditions.

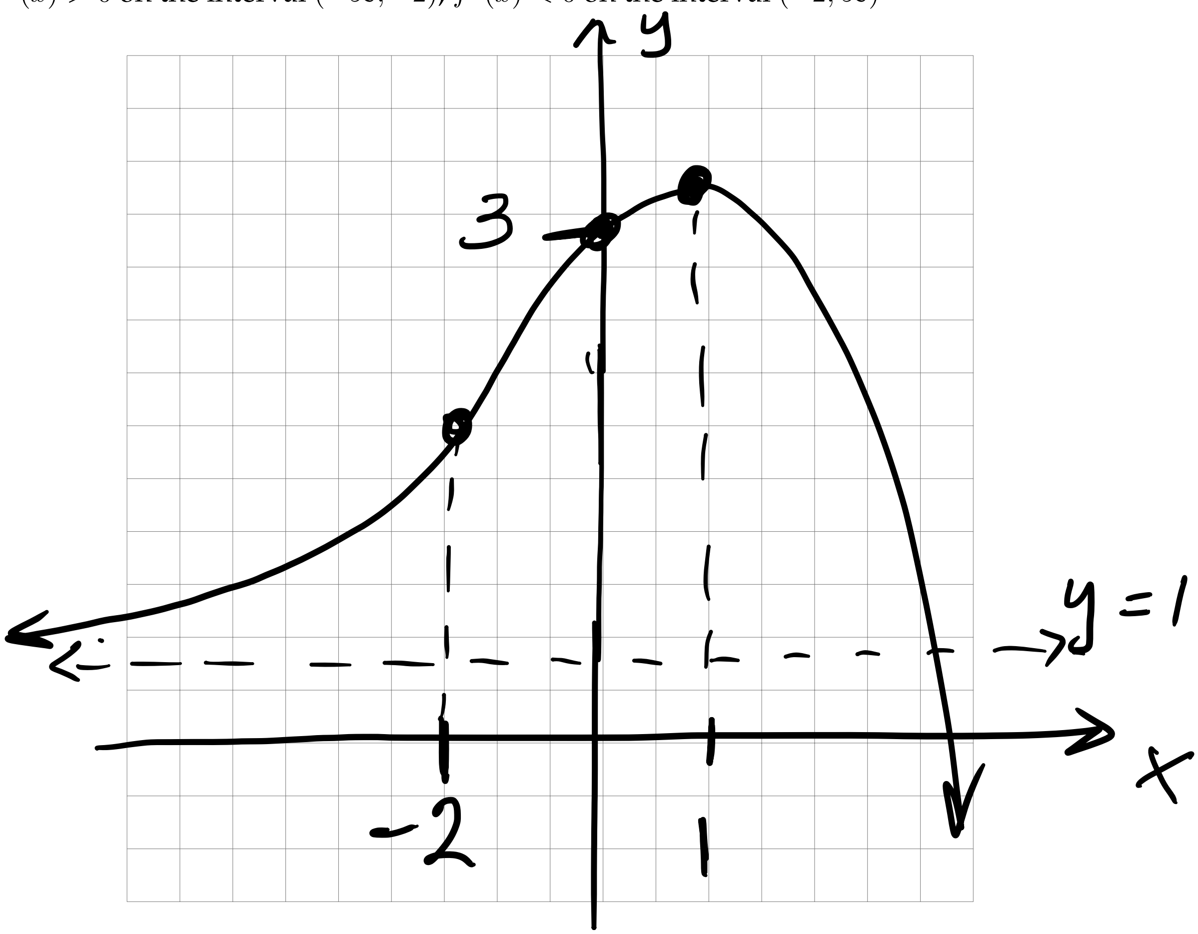
(a) The domain of $f(x)$ is $(-\infty, \infty)$.

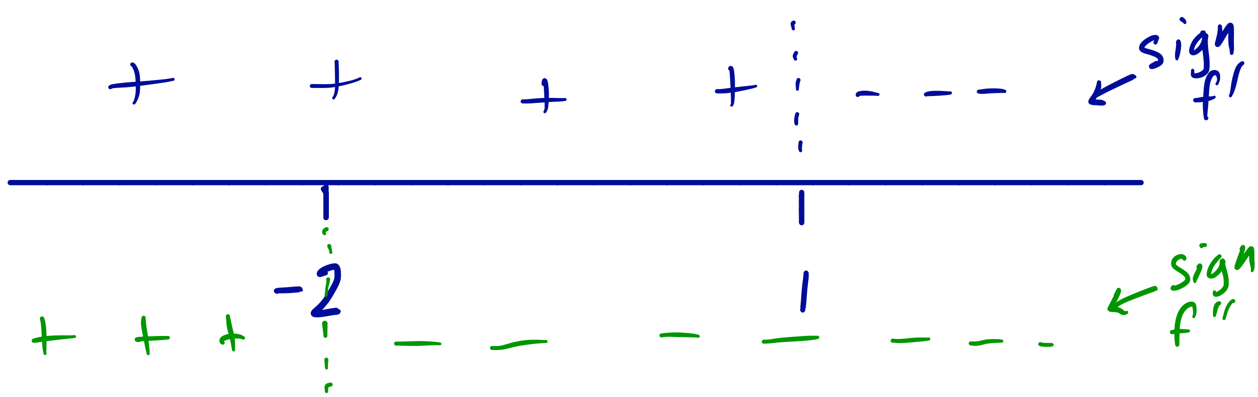
(b) $f(0) = 3$ (0,3) on graph

(c) $\lim_{x\to-\infty} f(x) = 1$ ← y=1 is a HA on the left.

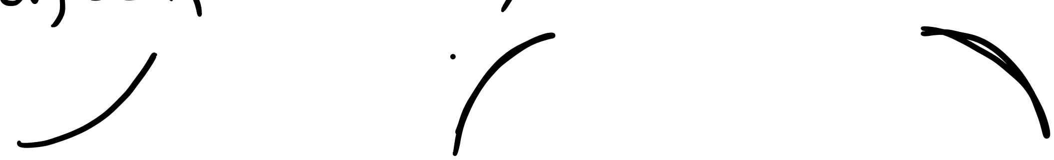
(d) $f'(x) > 0$ on the interval $(-\infty, 1)$; $f'(x) < 0$ on the interval $(1, \infty)$

(e) $f''(x) > 0$ on the interval $(-\infty, -2)$; $f''(x) < 0$ on the interval $(-2, \infty)$




incr, ccup　　incr, cc down　　decr cc down



$\boxed{8}$ (15 points) Use the information below to answer questions about the function $f(x)$. Make sure you answer the question!

$$f(x) = \frac{x^2}{x^2-1} + 3, \qquad f'(x) = \frac{-2x}{(x^2-1)^2}, \qquad f''(x) = \frac{2(3x^2+1)}{(x^2-1)^3}.$$

(a) Find the domain.

$(-\infty,-1)\cup(-1,1)\cup(1,\infty)$

(b) Determine the intervals on which the function is increasing/decreasing.

$f'=0$ when $x=0$

$f'$ undef when $x=\pm 1$. but not in domain!

$+ + \{ + + 0 - - \} - - -$ ← sign of $f'$

$-1 \quad 0 \quad 1$

$\frac{+}{+} \quad \frac{+}{+} \quad \frac{-}{+} \quad \frac{-}{+}$

Ans: $f$ is increasing on $(-\infty,-1)\cup(-1,0)$ and decreasing on $(0,1)\cup(1,\infty)$.

(c) Find the local maximum/minimum values of the function. If something doesn't exist, you must explicitly state this and justify your answer.

local maximum of $f(0)=3$.

no local minimum. only one critical number.

(d) Find the intervals of concavity.

$f''=0$ never

$f''$ undefined at $x=\pm 1$ not in domain

$+ + + \{ - - - \} + + +$ ← sign of $f''$

$-1 \quad 1$

ANS: $f$ is concave up on $(-\infty,-1)\cup(1,\infty)$ and concave down on $(-1,1)$.

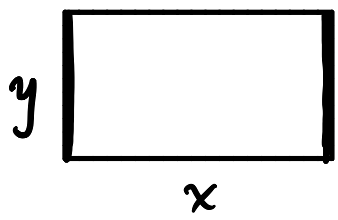
(e) Find the inflection points. If there aren't any, you must explicitly state this and justify your answer.

No inflection points.

Where the concavity changes, $f(x)$ is undefined.

9 (10 points) A rectangular plot of land is to be fenced in using two kinds of fencing. Two opposite sides will use heavy-duty fencing selling for \$3 a foot, while the remaining two sides will use standard fencing selling for \$2 a foot.

(a) What are the dimensions of the rectangular plot of greatest area that can be fenced in at a cost of \$360?




heavy duty fencing.

goal: maximize area, $A$.

$A = xy$

plug in for $y$

- Use cost info:

$\$360 = \$3 \cdot 2y + \$2 \cdot 2x$

So $360 = 6y + 4x$. So $y = \dfrac{360-4x}{6} = \boxed{60 - \frac{2}{3}x}$

Write area, $A$, as a function of ONE variable:

$A(x) = x \cdot \left(60 - \frac{2}{3}x\right) = 60x - \frac{2}{3}x^2$ ; domain $x > 0$.

$A'(x) = 60 - \frac{4}{3}x = 0$ ; So $x = \dfrac{3 \cdot 60}{4} = 45$.

If $x = 45$, $y = 60 - \frac{2}{3}(45) = 30$.

Answer: 45 ft by 30 ft, where the 30-ft-side is heavy duty.

(b) Use the First or Second Derivative Test to justify your conclusion in Part (a).

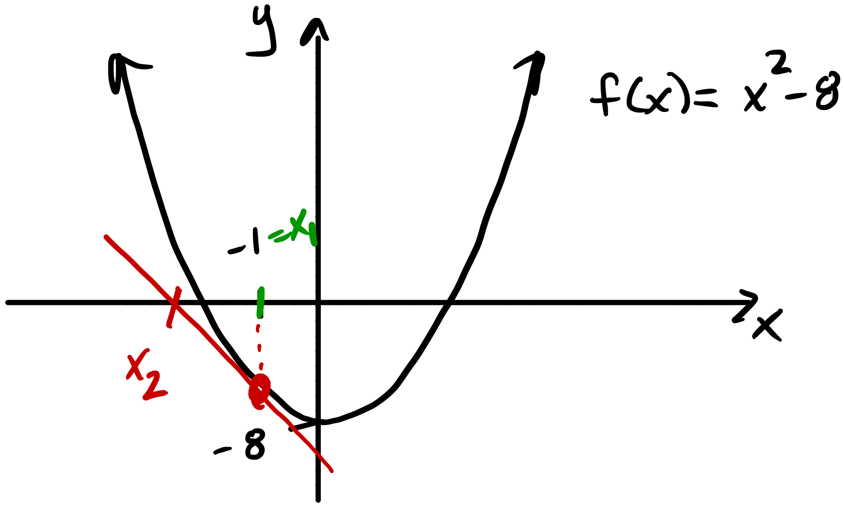
$A''(x) = -\frac{4}{3} < 0$. So $A(x)$ is concave down at $x = 45$, the only critical point. Thus, $A(x)$ has a maximum at $x = 45$.

10 (7 points) In this problem we are going to use Newton's method to estimate $\sqrt{8}$ using the function $f(x) = x^2 - 8$.

(a) State an appropriate initial value $x_1$ for use in applying Newton's method. Justify your answer.

Since $2^2 = 4 < 8$, $x=2$ is too small. Since $3^2 = 9 > 8$, $x=3$ is too big. So I choose an x-value between them, say $x = 2.5$.

(b) Sketch the function and illustrate the idea behind Newton's Method using starting point $x_1 = -1$.




(c) Suppose you are given an initial value of $x_1 = -1$. Find the next estimate $x_2$ given by Newton's method for a root of the function $f(x)$.

Newton's Formula : $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$.

So $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} = x_1 - \dfrac{(x_1)^2 - 8}{2x_1}$

using $x_1 = -1$, we obtain:

$$x_2 = (-1) - \frac{(-1)^2 - 8}{2(-1)} = -1 - \frac{-7}{-2} = -1 - \frac{7}{2} = \boxed{-\frac{9}{2}}$$