Your Name

End Time

Page	Total Points	Score
2	22	
3	10	
4	20	
5	16	
6	15	
7	12	
8	22	
9	20	
10	13	
Total	150	

- You will have 2.5 hours to complete the test.
- This test is closed notes and closed book and you may not use a calculator.
- In order to receive full credit, you must **show your work.** Please write out your computations on the exam paper.
- **PLACE A BOX AROUND** $\boxed{\textbf{YOUR FINAL ANSWER}}$ **to each question** where appropriate.

1. (22 pts) Find the following limits. If a limit does not exist, explain why and give $\pm\infty$ when appropriate.

(a) (3 pts) $\lim_{x\to 5} \frac{2x^2-50}{x^2-3x-10}$

$= \lim_{x\to 5} \frac{2(x^2-25)}{(x-5)(x+2)}$

$= \lim_{x\to 5} \frac{2(x-5)(x+5)}{(x-5)(x+2)}$

$= \lim_{x\to 5} \frac{2(x+5)}{(x+2)} = \boxed{\frac{20}{7}}$ ✓

(b) (3 pts) $\lim_{x\to 0} \frac{x}{\tan^{-1}(4x)}$ $\stackrel{H}{=} \lim_{x\to 0} \frac{1}{\left(\frac{4}{1+16x^2}\right)}$

($\frac{0}{0}$ form)

$= \boxed{\frac{1}{4}}$ ✓

(c) (3 pts) $\lim_{x\to-\infty} \frac{\sqrt{x^2+4x}}{4x+1}$

$= \lim_{x\to\infty} \frac{\sqrt{(-x)^2+4(-x)}}{4(-x)+1}$

$= \lim_{x\to\infty} \frac{\sqrt{x^2-4x}}{-4x+1} \cdot \frac{1/x}{1/x}$

$= \lim_{x\to\infty} \frac{\sqrt{1-4/x}}{-4+1/x} = \boxed{\frac{-1}{4}}$ ✓

(d) (3 pts) $\lim_{x\to 5^+} \frac{2-x^2}{x-5} = \boxed{-\infty}$ ✓

as $x\to 5^+$, $2-x^2 \to 2-25=-23$

as $x\to 5^+$, $x-5 \to$ a small pos. #.

we have a negative number divided by a small positive...

(e) (5 pts) $\lim_{x\to 2} \frac{x-\sqrt{3x-2}}{x^2-4} \cdot \frac{x+\sqrt{3x-2}}{(x+\sqrt{3x-2})}$

$= \lim_{x\to 2} \frac{x^2-(3x-2)}{(x+2)(x-2)(x+\sqrt{3x-2})}$

$= \lim_{x\to 2} \frac{x^2-3x+2}{(x+2)(x-2)(x+\sqrt{3x-2})}$

$= \lim_{x\to 2} \frac{(x-2)(x-1)}{(x+2)(x-2)(x+\sqrt{3x-2})}$

$= \lim_{x\to 2} \frac{x-1}{(x+2)(x+\sqrt{3x-2})}$

$= \frac{1}{4(2+\sqrt{6-2})}$

$= \boxed{\frac{1}{16}}$ ✓

(f) (5 pts) $\lim_{x\to\infty} (e^x+x)^{1/x}$

$y = (e^x+x)^{1/x}$

$\ln y = \ln(e^x+x)^{1/x} = \frac{\ln(e^x+1)}{x}$

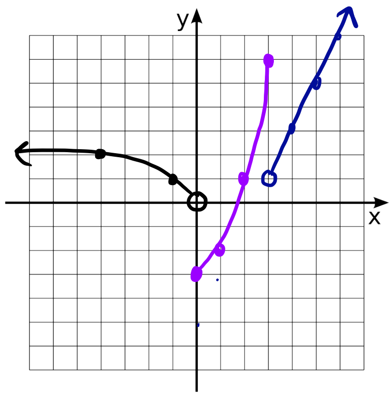
$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln(e^x+1)}{x}$

$\stackrel{H}{=} \lim_{x\to\infty} \frac{\left(\frac{e^x}{e^x+1}\right)}{1}$

$= 1$

Thus $\lim_{x\to\infty} \ln y = 1$ and $\boxed{\lim_{x\to\infty} y = e}$ ✓

2. (6 pts) Sketch the graph of $f(x) = \begin{cases} \sqrt{-x} & \text{if } x < 0 \\ x^2 - 3 & \text{if } 0 \le x \le 3 \\ 2x - 5 & \text{if } x > 3 \end{cases}$. Then, find the numbers at which $f(x)$ is discontinuous. Specify whether f is continuous from the right, from the left, or neither at any discontinuities.



discontinuous at $x=0$ and $x=3$.

continuous from left at $x=0$

continuous from right at $x=3$

3. (4 pts) Use the definition of the derivative to find $f'(x)$ if $f(x) = 7 + 5x - 3x^2$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{7+5(x+h) - 3(x+h)^2 - (7+5x-3x^2)}{h}$$

$$= \lim_{h\to 0} \frac{7 + 5x + 5h - 3x^2 - 6xh - 3h^2 - 7 - 5x + 3x^2}{h}$$

$$= \lim_{h\to 0} \frac{5h - 6xh - 3h^2}{h}$$

$$= \lim_{h\to 0} (5 - 6x - 3h)$$

$$= \boxed{5 - 6x}$$

4. (10 pts) Find and simplify the derivatives of the following functions.

a) $y = t^2 \arctan(2t)$

$y' = 2t \arctan(2t) + t^2 \cdot \frac{2}{1+(2t)^2}$

$= \boxed{2t\arctan(2t) + \frac{2t^2}{1+4t^2}}$

b) $f(x) = \frac{e^{1/x}}{x^2}$

$f'(x) = \frac{x^2 \cdot e^{1/x} \cdot (-1x^{-2}) + e^{1/x} \cdot 2x}{(x^2)^2}$

$= \boxed{\frac{e^{1/x}(-1+2x)}{x^4}}$

5. (10 pts) Find and simplify the derivatives of the following functions.

a) $y = \ln\left(\frac{\sqrt{x}}{x^2+1}\right)$

$y = \frac{1}{2}\ln x - \ln(x^2+1)$

$\boxed{y' = \frac{1}{2x} - \frac{2x}{x^2+1}}$

$y' = \frac{x^2+1 - 2x \cdot 2x}{2x(x^2+1)}$

$\boxed{y' = \frac{1-x^2}{2x(x^2+1)}}$

b) $g(x) = \tan^2(\sin 5x)$

$g'(x) = 2\tan(\sin(5x))\sec^2(\sin(5x)) \cdot 5\cos(5x)$

$= \boxed{10\tan(\sin(5x))\sec^2(\sin(5x))\cos(5x)}$

6. (9 pts) Find the derivatives of the following functions.

(c) (5 pts) $y = (\cos x)^{2x}$

$\ln y = 2x \ln(\cos x)$

$\frac{y'}{y} = 2\ln(\cos x) + 2x \cdot \left(\frac{-\sin x}{\cos x}\right)$

$y' = (2\ln(\cos x) - 2x\tan x)\,y$

$\boxed{y' = (2\ln(\cos x) - 2x\tan x)(\cos x)^{2x}}$

(d) (4 pts) $F(x) = \int_{x^5}^{1} \frac{t^2}{\sqrt{1-t^4}}dt = -\int_1^{x^5} \frac{t^2}{\sqrt{1-t^4}} dt$

$F'(x) = \frac{d}{dx}\left(-\int_1^{x^5} \frac{t^2}{\sqrt{1-t^4}} dt\right)$

$= \frac{-(x^5)^2}{\sqrt{1-(x^5)^4}} \cdot 5x^4 = \boxed{\frac{-5x^{14}}{\sqrt{1-x^{20}}}}$

7. (7 pts)

(a) (5 pts) Given $x^2 - xy + y^2 = 1$ find $\frac{dy}{dx}$.

$2x - xy' - y + 2yy' = 0$

$2yy' - xy' = y - 2x$

$y'(2y - x) = y - 2x$

$\boxed{y' = \frac{y-2x}{2y-x}}$

(b) (2 pts) Find an equation of the tangent line to $x^2 - 2xy + y^2 = 1$ at $(1, 0)$

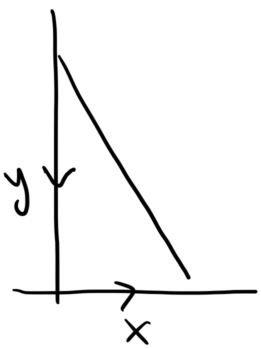
$m = \frac{0-2(1)}{2(0)-1} = 2$

$y - y_1 = m(x - x_1)$

$y - 0 = 2(x-1)$

$\boxed{y = 2x - 2}$

8. (5 pts) A ladder 10 feet long is leaning against the wall of a house. The base of the ladder is pulled away from the wall at a rate of 2 feet per second. How fast is the top of the ladder moving down the wall when it's base is 3 feet from the wall?



$x^2 + y^2 = 100$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$3(2) + \sqrt{91}\,\frac{dy}{dt} = 0$

$\boxed{\frac{dy}{dt} = -6/\sqrt{91} \text{ ft/sec}}$

know $dx/dt = 2$

want dy/dt when $x=3$

if $x=3$, $y=\sqrt{91}$

9. (4 pts) Find the absolute maximum and absolute minimum of $f(x) = x^3 - 6x^2 + 9x + 3$ on $[-1, 2]$

Step 1: find critical #s, make sure they are in domain

$f'(x) = 3x^2 - 12x + 9$

$0 = 3(x^2 - 4x + 3)$

$0 = 3(x-3)(x-1)$

$x=3$ (not in domain) $\quad x=1$

Step 2: Test CNs + endpts in f(x).

$f(-1) = -1 - 6 - 9 + 3 = -13$

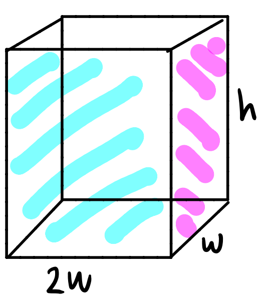
$f(1) = 1 - 6 + 9 + 3 = 7$

$f(2) = 8 - 24 + 18 + 3 = 5$

Abs min $f(-1) = -13$

Abs max $f(1) = 7$

10. (6 pts) An open-top box is to be made whose base length is twice its width. If 600 square centimeters is to be used in its construction, what dimensions maximize the volume of the box?



$V = 2w^2h$, $\quad A = 2w^2 + 2\cdot 2wh + 2\cdot wh$

$600 = 2w^2 + 6wh$

$\frac{600 - 2w^2}{6w} = h$

So, $V = 2w^2\left(\frac{600 - 2w^2}{6w}\right) = \frac{1}{3}w(600 - 2w^2)$

$V = 200w - \frac{2}{3}w^3$

$V' = 200 - 2w^2$

$0 = 200 - 2w^2$

$200 = 2w^2$

$w^2 = 100$

$w = 10$

$2w = 20$

$h = \frac{600 - 200}{60} = \frac{40}{6}$

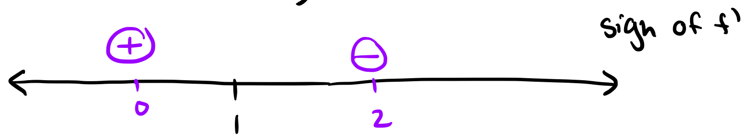
$\boxed{\text{Base } 10 \text{ cm} \times 20 \text{ cm}}$, $\boxed{h = \frac{20}{3}}$

11. (12 pts) Let $f(x) = xe^{-x}$

(a) (3 pts) Find the intervals of increase/decrease.

$$f'(x) = e^{-x} - xe^{-x}$$

$$0 = e^{-x}(1-x) \Rightarrow x = 1$$



$$\boxed{f \text{ inc on } (-\infty, 1) + \text{ dec on } (1, \infty)}$$

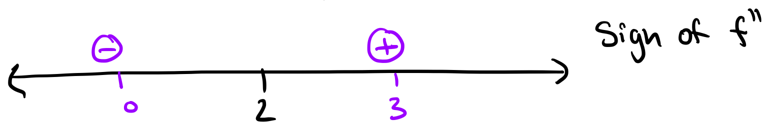
(b) (2 point) Find the local extrema. Classify as a maxima or minima.

$$\boxed{f(1) = e^{-1} \text{ is a local max}}$$

(c) (3 point) Find the intervals of concavity. $f'(x) = e^{-x} - xe^{-x}$

$$f''(x) = -e^{-x} - e^{-x} + xe^{-x}$$

$$0 = e^{-x}(x-2) \Rightarrow x = 2$$



$$\boxed{f \text{ is CU on } (2, \infty) + \text{ CD on } (-\infty, 2)}$$

(d) (2 point) Find any inflection points.

$$\boxed{f(2) = 2e^{-2} \text{ is POI}}$$

(e) (2 pt) Find the horizontal asymptotes, if any.

$$\lim_{x\to\infty} xe^{-x} = \lim_{x\to\infty} \frac{x}{e^x} \overset{H}{=} \lim_{x\to\infty} \frac{1}{e^x} = \boxed{0}$$

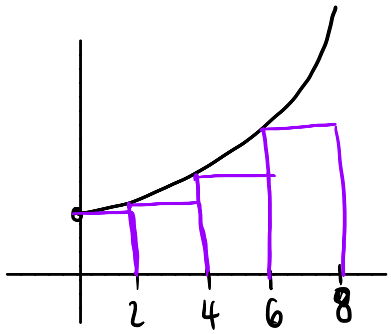
$$\lim_{x\to-\infty} xe^{-x} = \lim_{x\to+\infty} (-x)e^{-(-x)}$$

$$= \lim_{x\to\infty} (-xe^{x}) = \boxed{-\infty}$$

$$\boxed{\text{HA @ } y = 0}$$

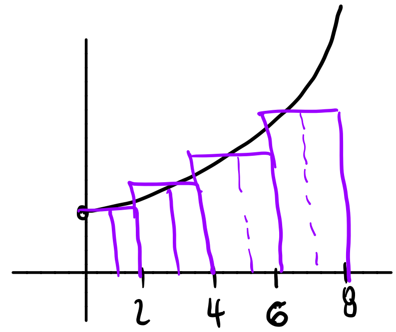
12. (6 pts) Given $f(x) = x^2 + 1$, estimate the area under the curve from $[0, 8]$ with $n = 4$ using:

a) Left endpoints.



$$L_4 = 2(f(0) + f(2) + f(4) + f(6))$$
$$= 2(1 + 5 + 17 + 37)$$
$$= 2(60) = \boxed{120}$$

b) Midpoints.



$$M_4 = 2(f(1) + f(3) + f(5) + f(7))$$
$$= 2(2 + 10 + 26 + 50)$$
$$= 2(88)$$
$$= \boxed{176}$$

13. (8 pts) Find the following indefinite integrals.

a) $\displaystyle\int \left(x - \frac{1}{x}\right)^2 dx$

$$= \int \left(x^2 - 2 \cdot x \cdot \frac{1}{x} + \frac{1}{x^2}\right) dx$$
$$= \int (x^2 - 2 + x^{-2})\, dx$$
$$= \boxed{\frac{1}{3}x^3 - 2x - \frac{1}{x} + C}$$

b) $\displaystyle\int \sec x(\sec x + \tan x) dx$

$$= \int (\sec^2 x + \sec x \tan x) dx$$
$$= \int \sec^2 x\, dx + \int \sec x \tan x\, dx$$
$$= \boxed{\tan x + \sec x + C}$$

14. (8 pts) Find the following indefinite integrals.

a) $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = \int \frac{e^u}{\sqrt{x}} \cdot 2\sqrt{x}\, du$

$u = \sqrt{x}$

$du = \frac{1}{2}x^{-1/2} dx$

$2\sqrt{x}\, du = dx$

$$= 2 \int e^u\, du$$
$$= 2e^u + C$$
$$= \boxed{2e^{\sqrt{x}} + C}$$

b) $\displaystyle\int \frac{x^3}{1 + x^4} dx = \int \frac{x^3}{u} \cdot \frac{1}{4x^3}\, du$

$u = 1 + x^4$

$du = 4x^3 dx$

$$= \frac{1}{4} \int \frac{1}{u}\, du$$
$$= \frac{1}{4} \ln|u| + C$$
$$= \boxed{\frac{1}{4} \ln(1 + x^4) + C}$$

15. (10 pts) Find the following indefinite integrals.

a) $\int \frac{x}{(x-3)^2}dx = \int \frac{u+3}{u^2}\,du$

$u = x-3$
$du = dx$
$x = u+3$

$= \int (\frac{1}{u} + 3u^{-2})\,du$

$= \ln|u| + \frac{3u^{-1}}{-1} + C$

$= \boxed{\ln|x-3| - \frac{3}{x-3} + C}$

b) $\int \frac{x+4}{x^2+1}dx = \int \frac{x}{x^2+1} + \frac{4}{x^2+1}\,dx$

$u = x^2+1$
$du = 2x\,dx$

$= \int \frac{x}{x^2+1}dx + \int \frac{4}{x^2+1}\,dx$

$= \int \frac{x}{u}\frac{du}{2x} + 4\tan^{-1}x + C$

$= \frac{1}{2}\ln|u| + 4\arctan x + C$

$= \boxed{\frac{1}{2}\ln|x^2+1| + 4\arctan x + C}$

16. (10 pts) Find the following definite integrals.

a) $\int_1^9 \frac{\sqrt{u}-2u^2}{u}du = \int_1^9 (\frac{\sqrt{u}}{u} - \frac{2u^2}{u})\,du$

$= \int_1^9 (u^{-1/2} - 2u)\,du$

$= (2\sqrt{u} - u^2)\Big|_1^9$

$= 6 - 81 - (2-1)$

$= 6 - 81 - 1$

$= 6 - 82$

$= \boxed{-76}$

b) $\int_0^{\pi/4} \sec^2 x \tan^2 x\,dx = \int_0^1 u^2\,du$

$u = \tan x$
$du = \sec^2 x\,dx$
$x = 0,\ u = \tan 0 = 0$
$x = \pi/4,\ u = \tan\frac{\pi}{4} = 1$

$= \frac{1}{3}u^3\Big|_0^1$

$= \boxed{\frac{1}{3}}$

17. (5 pts) A particle is moving with the given data. Find the position of the particle. $a(t) = 2t + \sin t + 1$, $s(0) = 1$, $v(0) = -2$

$v(t) = t^2 - \cos t + t + C$

$-2 = 0 - \cos 0 + 0 + C$

$-2 = -1 + C \Rightarrow C = -1$

$v(t) = t^2 - \cos t + t - 1$

$s(t) = \frac{1}{3}t^3 - \sin t + \frac{1}{2}t^2 - t + D$

$s(0) = 1$ so,
$1 = 0 - 0 + 0 - 0 + D$

$\boxed{s(t) = \frac{1}{3}t^3 - \sin t + \frac{1}{2}t^2 - t + 1}$

18. A particle moves along the x-axis with velocity function $v(t) = t^2 - 4$, where v is measured in meters per second.

(a) (3 pts) Find the displacement over the time interval $[0, 3]$

$\text{disp} = \int_0^3 (t^2 - 4)\,dt$

$= \left(\frac{1}{3}t^3 - 4t\right)\Big|_0^3$

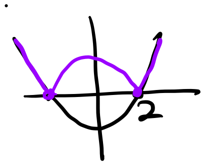
$= \frac{1}{3}(27) - 12$

$= 9 - 12$

$= \boxed{-3 \text{ m}}$

backwards 3 m from start position.

(b) (5 pts) Find the distance traveled over the time interval $[0, 3]$.



$\text{dist} = \int_0^3 |t^2 - 4|\,dt$

$= -\int_0^2 (t^2 - 4)\,dt + \int_2^3 (t^2 - 4)\,dt$

$= \left(-\frac{1}{3}t^3 + 4t\right)\Big|_0^2 + \left(\frac{1}{3}t^3 - 4t\right)\Big|_2^3$

$= -\frac{8}{3} + 8 + 9 - 12 - (8/3 - 8)$

$= -\frac{16}{3} + 16 - 3$

$= -\frac{16}{3} + 13 = -\frac{16}{3} + \frac{39}{3} = \boxed{\frac{23}{3} \text{ m}}$