Math F251 Final Exam Fall 2021

Name: ______________________

Section: ☐ **F01 (Faudree)**
☐ **F02 (Gossell)**
☐ **UX1 (Gossell)**

Rules:

You have 2 hours to complete the exam.

Partial credit will be awarded, but you must show your work.

No other aids are permitted.

Place a box around your $\boxed{\text{FINAL ANSWER}}$ to each question where appropriate.

Turn off anything that might go beep during the exam.

Problem	Possible	Score
1	8	
2	8	
3	10	
4	8	
5	10	
6	10	
7	6	
8	6	
9	5	
10	5	
11	12	
12	12	
Extra Credit	5	
Total	100	

1. (8 points)

Find the derivative of each of the following functions. You do not need to simplify your answer.

a. $f(x) = (\cos x)(\ln(x^2 + 1))$

$f'(x) = -\sin(x)\ln(x^2+1) + (\cos x)\left(\frac{2x}{x^2+1}\right)$

$\leftarrow e^{x^{\frac{1}{2}}}$

b. $g(x) = e^{\sqrt{x}} + 5x^3 + \sin\left(\frac{\pi}{4}\right)$

$g'(x) = \frac{1}{2}x^{-1/2}\, e^{\sqrt{x}} + 15x^2 + 0$

2. (8 points)

Evaluate the following indefinite integrals.

a. $\int \left(2x^4 - \frac{4}{x^2}\right) dx = \int (2x^4 - 4x^{-2})dx = \frac{2}{5}x^5 + 4x^{-1} + C$

b. $\int \frac{\sec^2 x}{\tan x} dx = \ln|\tan(x)| + C$

3. (10 points)

Evaluate the following limits. If you use L'Hopital's Rule, please indicate the form of the limit (0/0 or ∞/∞).

* Note: We meant this to be: $\lim_{x\to 6} \frac{\frac{1}{6}-\frac{1}{x}}{x^2-36}$

* **a.** $\lim_{x\to 6} \frac{\frac{1}{6}-\frac{1}{x}}{x-36}$. $= \frac{\frac{1}{6}-\frac{1}{6}}{6-36} = \frac{0}{30} = \boxed{0}$

b. $\lim_{x\to 3^-} \frac{2x^2-3x}{x^2-7x+12} = \lim_{x\to 3^-} \frac{x(2x-3)}{(x-3)(x-4)} = \boxed{+\infty}$

$2\cdot 9-6=12$

$9-21+12=0$

form $\frac{12}{0}$. So the limit will be infinite. Determine +/- Sign.

As $x\to 3^-$, $x^2-7x+12 \to 0^+$ and $x(2x-3)\to 12>0$. So the quotient is positive.

4. (8 points)

The temperature in a cabin is given by

$$T(t) = 55 + \frac{20t}{t+1}$$

where T is measured in degrees Fahrenheit and $t \geq 0$ is measured in minutes after starting the wood stove.

a. At what **rate** is the temperature changing at time $t = 0$? Include units in your answer.

Find $T'(0)$.

$$T'(t) = 0 + \frac{(t+1)(20)-20t(1)}{(t+1)^2} = \frac{20}{(t+1)^2}.$$

So, $T'(0) = 20\ {}^\circ F/min$

b. Compute $\lim_{t\to\infty} T(t)$ and explain what this number means in language the general public might understand.

$$\lim_{t\to\infty}\left(55+\frac{20t}{t+1}\right) = 55+20 = \boxed{75}$$

Interpretation: As time goes on, long-term the temperature in the cabin approaches 75°F.

$\leftarrow h(0)=2$

5. (10 points)

A drone is launched off a 2-foot platform. Its upward velocity in feet per second at t seconds is measured by the function $v(t) = \frac{3}{1+t^2}$. $\leftarrow 3(1+t^2)^{-1}$

a. Find $h(t)$, the height of the drone at t seconds.

$$h(t) = \int v(t)\,dt$$

$$= \int \frac{3}{1+t^2}\,dt = 3\arctan(t)+C$$

$$2 = h(0) = 3\arctan(0)+C = C$$

So $C=2$

$$\boxed{h(t) = 3\arctan(t)+2}$$

b. Find $h(1)$, the height after 1 second. Include units. Hint: $\tan(\pi/4) = 1$.

$$h(1) = 3\arctan(1)+2 = \boxed{\frac{3\pi}{4}+2 \text{ feet}}$$

c. Find $a(t)$, the acceleration function at t seconds.

$$a(t) = v'(t) = 3(-1)(1+t^2)^{-2}(2t) = \boxed{\frac{-6t}{(1+t^2)^2}}$$

d. Find $a(1)$, the acceleration after 1 second. Include units.

$$a(1) = \frac{-6}{(2)^2} = \frac{-6}{4} = \boxed{\frac{-3}{2} \text{ ft/s}^2}$$

e. Find $v(1)$, the upward velocity after 1 second. Include units.

$$v(1) = \frac{3}{1+1^2} = \boxed{\frac{3}{2} \text{ ft/s}}$$

f. Use your answers from parts (d) and (e) above to determine whether the drone speeding up or slowing down when $t = 1$.

$\boxed{\text{Slowing down}}$

(Because $a(1)$ and $v(1)$ have different signs.)

6. (10 points)

A manufacturer discovers that the revenue gained by producing and selling x products is $R(x) = 420x$ and the cost is $C(x) = 2x^2 + 5000$.

a. Write the profit function, $P(x)$. (Hint: Remember that profit is revenue minus cost: $P(x) = R(x) - C(x)$.)

$$P(x) = 420x - (2x^2+5000) = \boxed{420x - 2x^2 - 5000}$$

b. What is the domain of the profit function?

$[0, \infty)$ or $(0, \infty)$

c. How many products should the manufacturer produce to maximize the profit? Be sure to justify that your answer is correct. That is, use Calculus to show that your answer indeed does represent a maximum or minimum.

$$P'(x) = 420 - 4x = 0$$

$$x = \frac{420}{4} = 105$$

* Check it's a max:

$+ \; + \; + \quad 0 \quad - \; - \; -$ ← Sign P'

$0 \quad 1 \quad 105 \quad 1000$

By the first derivative test, P is maximized when the manufacturer produces and sells 105 products.

* Note: You could use the 2nd derivative test: $P'' = -4 < 0 \Rightarrow$ max. You could observe that P is a downward opening parabola.

7. (6 points)

The volume of a spherical balloon is given by $V = \frac{4}{3}\pi r^3$. If the balloon is being inflated at a rate of 32π cubic inches per minute, how fast is the balloon's radius changing when the radius is 2 inches? Give units with your answer.

$$\frac{dV}{dt} = 32\pi \text{ in}^3/\text{min}$$

Find $\frac{dr}{dt}$ when $r = 2$ in

If $V = \frac{4}{3}\pi r^3$, then

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

plug in & solve:

$$32\pi = 4\pi \cdot 2^2 \frac{dr}{dt}$$

So $\frac{dr}{dt} = \frac{32}{16} = \underline{2 \text{ in/min}}$

8. (6 points)

The function $G(x)$ is continuous on its domain $(-\infty, 0) \cup (0, \infty)$.

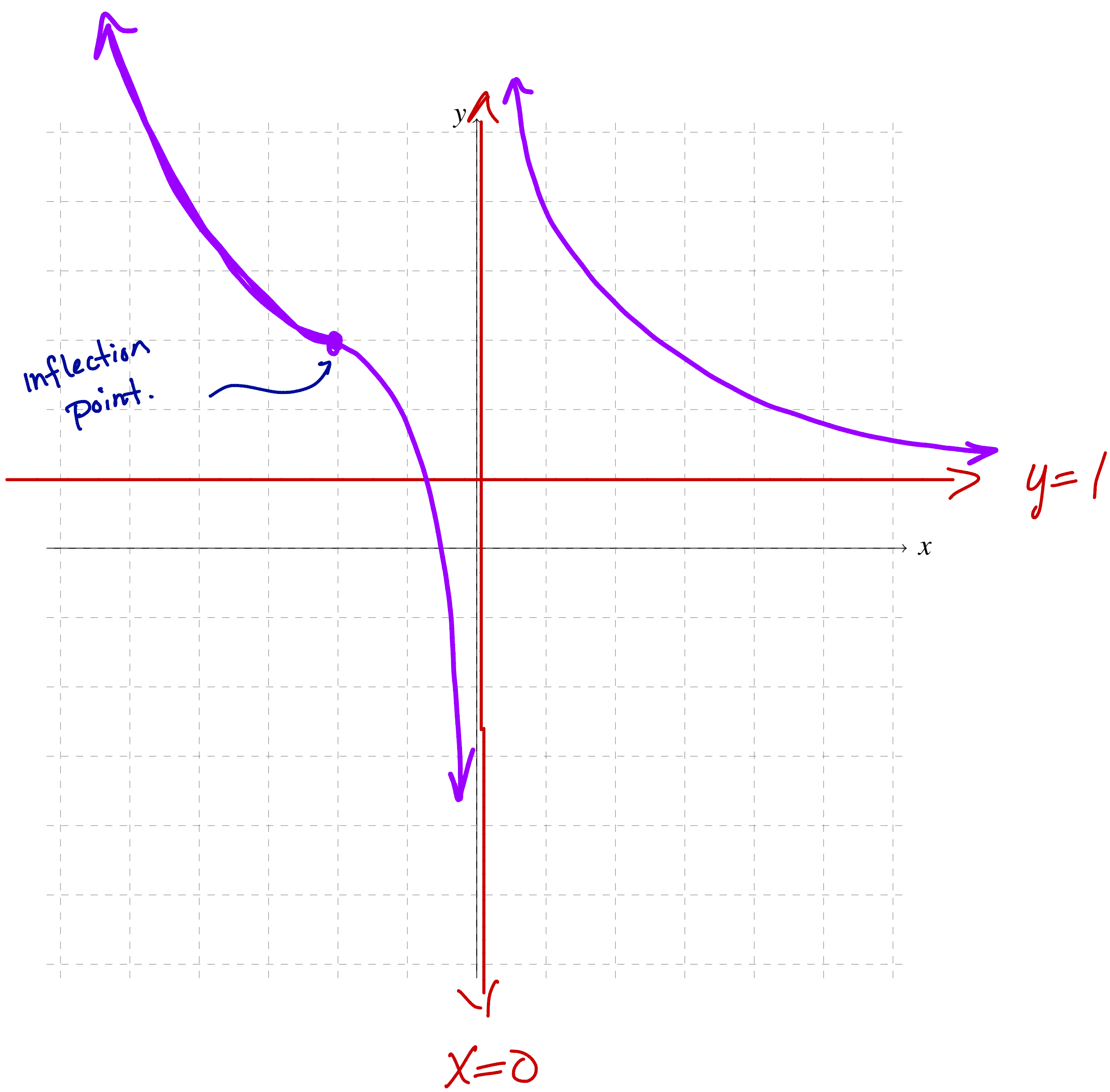
- $G'(x)$ is negative for all x in the domain $(-\infty, 0) \cup (0, \infty)$. $\mapsto$ G decreases
- $G''(x)$ is negative in the interval $(-2, 0)$
- $G''(x)$ is positive in the interval $(-\infty, -2) \cup (0, \infty)$.

cc up   cc down   cc up

-2   0

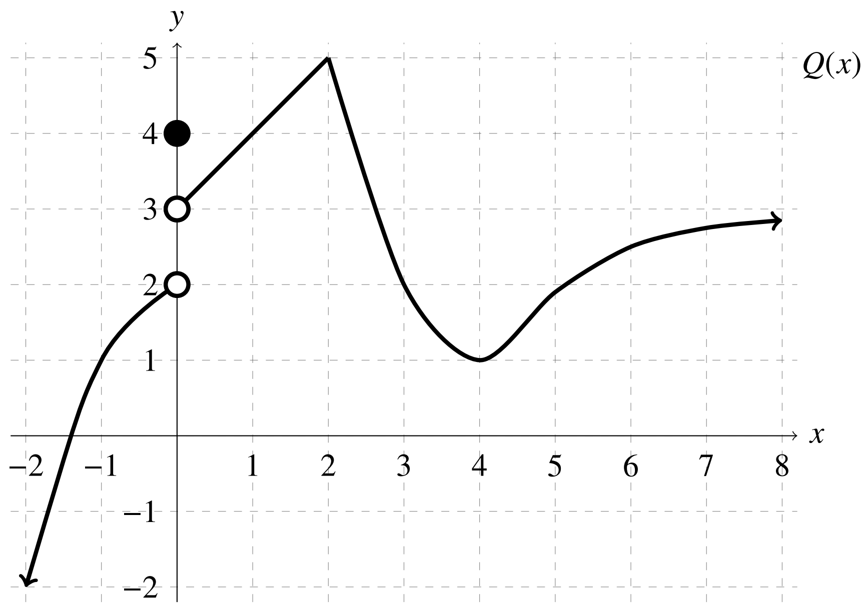
- $\lim_{x\to\infty} G(x) = 1$. $\mapsto$ HA at $x=1$
- $\lim_{x\to 0^+} G(x) = \infty$ and $\lim_{x\to 0^-} G(x) = -\infty$ VA at $x=0$

Sketch the graph of $G(x)$.



9. (5 points)

Consider the function $Q(x)$ graphed below.



a. Find $\lim_{x\to 0^-} Q(x)$ 2

b. Find $\lim_{x\to 4} Q(x)$ 1

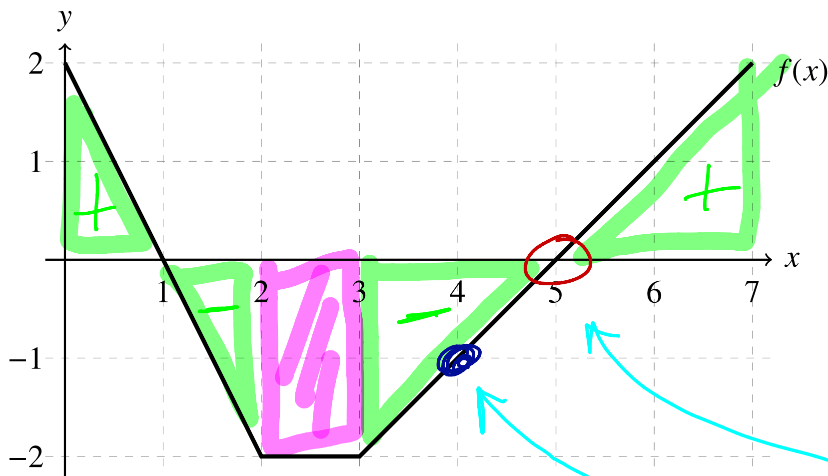
c. Find $\lim_{x\to \infty} Q(x)$ 3

d. For what values of x, if any, does $Q(x)$ fail to be continuous? $x=0$

e. At what values of x, if any, does $Q'(x)$ not exist? $x=0,\ x=2$

10. (5 points)

Consider the function $f(x)$ graphed below.



a. What is the value of $f(1)$? 0

b. What is the value of $f'(1)$? -2

The following questions concern $H(x) = \int_0^x f(s)\,ds$.

c. What is the value of $H(7)$?

$$\int_0^7 f(x)\,dx = -2 \quad \text{or} \quad = \underbrace{1-1-2+2-2} = -2$$

d. What is the value of $H'(4)$?

$$H'(4) = f(4) = -1$$

e. At what values of x does $H(x)$ have a local minimum?

$H' = f$. Where f changes from $-$ to $+$:

at $x = 5$

11. (12 points)

The following questions concern $f(x) = \dfrac{x^2-1}{(x-3)^2}$. Note $f'(x) = -\dfrac{2(3x-1)}{(x-3)^3}$ and $f''(x) = \dfrac{12(x+1)}{(x-3)^4}$.

a. Find any critical points of $f(x)$.

Set $f'=0$. $2(3x-1)=0$. So $x=1/3$.

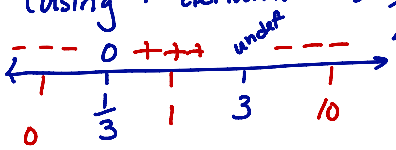
f' is undefined when $x=3$.

But $x=3$ isn't in the domain of $f(x)$.

Answer: $x=\frac{1}{3}$

b. Identify the locations of any local minimums or local maximums. Justify your conclusions. If no local minimum or local maximum exists, state this explicitly.

(Using 1st derivative test)



← Sign of f'

$f'(0) = \dfrac{-+-}{-} = -$

$f'(1) = \dfrac{-++}{-} = +$

$f'(10) = \dfrac{-++}{+} = -$

Answer: f has a local min at $x=1/3$ and no local max.

(using 2nd derivative test)

$f''(\frac{1}{3}) = \dfrac{12(1\frac{1}{3})}{(\frac{1}{3}-3)^4} = \dfrac{+}{+} = + > 0$

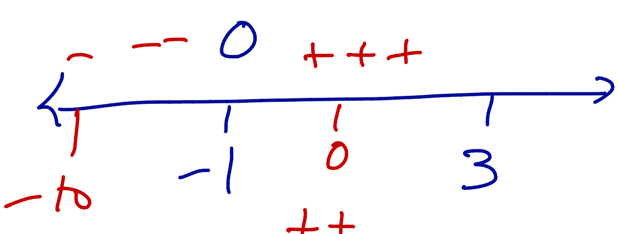
So $f(x)$ has a min at $x=\frac{1}{3}$.

There is no local max since f has no other critical point.

(Note $f''(3)=$ undef isn't helpful alone)

c. Does $f(x)$ have any inflection points? Justify your conclusions.

$f''=0$ when $x=-1$ ← Note: This alone is not sufficient.



← Sign of f''

-10: $\dfrac{+-}{+}$

0: $\dfrac{++}{+}$

f has an inflection point at $x=-1$ because f'' changes signs here.

(Note f does not have an inflection point at $x=3$ because $x=3$ isn't in the domain and the sign of f'' does not change signs at $x=3$.)

12. (12 points)

Water flows into a tank at a rate of $r(t) = 6 + 4x - x^2$ liters per minute from $t = 0$ to $t = 5$ minutes.

a. Compute $\int_0^2 r(t)\,dt$.

$$\int_0^2 r(t)\,dt = \int_0^2 (6+4t-t^2)\,dt = 6t+2t^2-\frac{1}{3}t^3\Big]_0^2 = \left(12+8-\frac{8}{3}\right)-(0)$$

$$= 20-\frac{8}{3} = \frac{52}{3}$$

b. Interpret your answer from part (a) in the context of the problem. Make sure to include units.

The net change of the volume of water in the tank between the start ($t=0$) and 2 minutes later ($t=2$) is $\frac{52}{3}$ liters.

c. At time $t = 0$, the tank contained 50 liters of water. How much water is in the tank at time $t = 2$?

$$50 + \frac{52}{3} \text{ liters} = \frac{202}{3} \text{ liters}$$

13. (Extra Credit: 5 points)

Evaluate $\lim_{x\to 0^+} \tan x \ln x$. You must show your work.

$$\lim_{x\to 0^+} \tan x \ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{\cot(x)} \overset{H}{=} \lim_{x\to 0} \frac{\frac{1}{x}}{-\csc^2 x} = \lim_{x\to 0} \frac{-\sin^2 x}{x}$$

form $0\cdot\infty$ form $\frac{0}{0}$ form $\frac{0}{0}$

$$\overset{H}{=} \lim_{x\to 0^+} \frac{-2\sin x\cos x}{1} = \frac{-2\cdot 0\cdot 1}{1} = 0$$

12. (12 points)

Water flows into a tank at a rate of $r(t) = 6 + 4x - x^2$ liters per minute from $t = 0$ to $t = 5$ minutes.

a. Compute $\int_0^2 r(t)\, dt$.

b. Interpret your answer from part (a) in the context of the problem. Make sure to include units.

c. At time $t = 0$, the tank contained 50 liters of water. How much water is in the tank at time $t = 2$?

13. (Extra Credit: 5 points)

Evaluate $\lim_{x\to 0^+} \tan x \ln x$. You must show your work.