

Name: Solutions**Rules:**

You have 90 minutes to complete the exam.

Partial credit will be awarded, but you must show your work.

You may have a single handwritten  $3 \times 5$  notecard.

Calculators are not allowed.

Place a box around your **FINAL ANSWER** to each question where appropriate.

Turn off anything that might go beep during the exam.

Good luck!

Problem	Possible	Score
1	10	
2	10	
3	10	
4	10	
5	18	
6	10	
7	10	
8	10	
9	12	
Extra Credit	5	
Total	100	

1. (10 points)

- (a) Find the linear approximation (also known as the linearization) of the function
- $f(x) = \sqrt{x}$
- when
- $a = 1$
- .

$$f(x) = x^{\frac{1}{2}}; f(1) = 1$$

$$f'(x) = \frac{1}{2} x^{-\frac{1}{2}}; f'(1) = \frac{1}{2} = 0.5$$

point (1,1)

$$\text{slope} = m = 0.5 \\ = \frac{1}{2}$$

$$y - 1 = 0.5(x - 1)$$

$$y = 1 + 0.5(x - 1)$$

ANSWER:

$$L(x) = 1 + 0.5(x - 1) \text{ OR}$$

$$L(x) = 1 + \frac{1}{2}(x - 1)$$

- (b) Use the linear approximation from part (a) to estimate
- $\sqrt{1.05}$
- . Your answer must be in the form of a simplified decimal or an exact fraction.

decimals  $\sqrt{1.05} \approx L(1.05) = 1 + 0.5(1.05 - 1) = 1 + (0.5)(0.05) \\ = 1 + 0.025 = 1.025$

fractions  $\sqrt{1.05} \approx L(1.05) = 1 + \frac{1}{2}(1.05 - 1) = 1 + \frac{1}{2}(0.05) = 1 + \frac{1}{2}\left(\frac{5}{100}\right) = 1 + \frac{5}{200} \\ = \frac{205}{200} = \frac{41}{40}$

2. (10 points) Evaluate the following limits.
- You must show your work and justify your answer to earn full credit.**
- If you apply L'Hopital's Rule, you should indicate this, by writing
- $\stackrel{H}{=}$
- or
- $\stackrel{L'H}{=}$
- or some other clear indication.

$$(a) \lim_{x \rightarrow \infty} \frac{(\sqrt{x^2 + 4}) \left(\frac{1}{x}\right)}{(6x + 4) \left(\frac{1}{x}\right)} = \lim_{x \rightarrow \infty} \frac{\sqrt{\frac{x^2 + 4}{x^2}}}{6 + \frac{4}{x}} = \lim_{x \rightarrow \infty} \frac{\sqrt{1 + \frac{4}{x^2}}}{6 + \frac{4}{x}} = \frac{1}{6}$$

$$(b) \lim_{x \rightarrow 0} \frac{x^2}{4 - 4 \cos(x)} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{2x}{4 \sin(x)} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{2}{4 \cos(x)} = \frac{2}{4} = \frac{1}{2}$$

form  $\frac{0}{0}$  form  $\frac{0}{0}$  using  $\cos(0) = 1$

3. (10 points) The formula for the volume,  $V$ , of a cone in terms of its radius  $r$  and height  $h$  is  $V = \frac{1}{3}\pi r^2 h$ . If the volume of the cone remains **constant** and the radius of the cone is increasing at a rate of 5 cm/s, determine the rate of change of the height of the cone at the instant the radius is 10 cm and the height is 6 cm. **Interpret your answer using a complete sentence.**

$$V = \frac{1}{3}\pi r^2 h$$

$$\frac{dV}{dt} = 0$$

$$\frac{dr}{dt} = 5 \text{ cm/s}$$

$$\text{Find } \frac{dh}{dt} \text{ when } r=10, h=6$$

Set up

$$\frac{dV}{dt} = \left(\frac{1}{3}\pi\right) \left(2r \frac{dr}{dt} h + r^2 \frac{dh}{dt}\right)$$

Take derivative.  
- wrt time  $t$   
- use prod. rule

$$0 = \left(\frac{\pi}{3}\right) \left(2(10)(5)(6) + 10^2 \frac{dh}{dt}\right)$$

Plug in

$$-100 \frac{dh}{dt} = 600$$

$$\frac{dh}{dt} = -6 \text{ cm/s}$$

Solve for  $\frac{dh}{dt}$

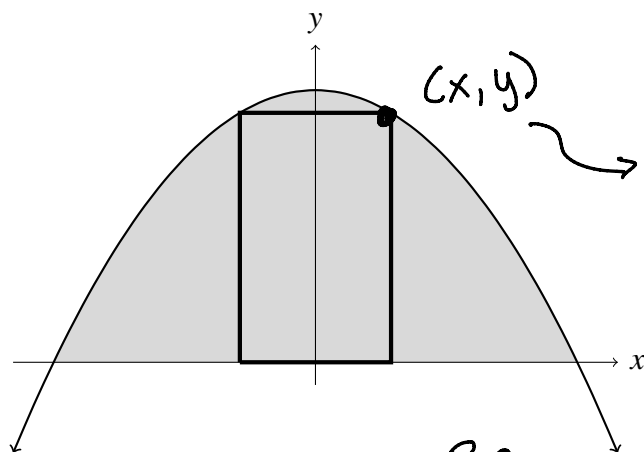
The height is decreasing at a rate of 6 cm/s.

answer

get direction correct

get units correct

4. (10 points) Determine the maximum area of a rectangle with base on the  $x$ -axis inscribed between the parabola  $y = 12 - x^2$  and the  $x$ -axis (See figure below.) Note: Your solution must use Calculus to **justify** that your answer is correct.



goal: maximize area

$$y = 12 - x^2$$

$$A = \text{area of rectangle} = 2xy$$

answer: Maximum area is 32

Need Area,  $A$ , as a function of 1 variable.

So substitute in for  $y$ .

$$A(x) = 2x(12 - x^2) = 2(12x - x^3); \text{ domain } (0, \sqrt{12})$$

There are many correct answers, here.

Find critical points:

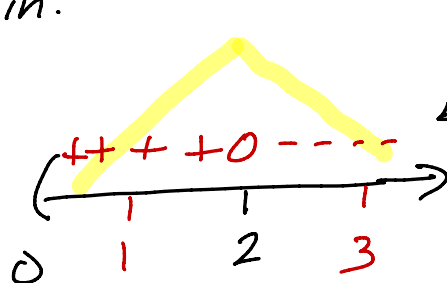
$$A'(x) = 2(12 - 3x^2) = 0. \text{ So } 12 - 3x^2 = 0 \text{ or } x^2 = 4, x = \pm 2$$

only  $x = +2$  is in my domain.

Check it's a maximum!

First Derivative Test:

So  $A(x)$  has a maximum at  $x = 2$ .



$$A'(1) = (2)(12 - 3) > 0$$

$$A'(3) = (2)(12 - 27) < 0$$

Plug  $x = 2$  into  $A(x)$  to find the maximum area:

$$A(2) = 2 \cdot 2(12 - 2^2) = 4(8) = 32$$

5. (18 points) Use the information below to answer questions about the function  $f(x)$ . You must show your work to earn full credit.

$$f(x) = \frac{x}{e^x}, \quad f'(x) = \frac{-(x-1)}{e^x}, \quad f''(x) = \frac{x-2}{e^x}.$$

- (a) Determine the intervals on which  $f(x)$  is increasing/decreasing.

$f' = 0$  when  $x=1$ ,  $f'$  never undefined

++++ 0 ---- ← Sign of  $f'$

← 0 1 2 →

$f'(0) = \frac{(-)(-)}{+} > 0$

$f'(2) = \frac{(-)(+)}{-} < 0$

ANSWER:

$f(x)$  is increasing on  $(-\infty, 1)$  and decreasing on  $(1, \infty)$

- (b) Find the  $x$ -values that correspond to any local maximums or local minimums of  $f(x)$ .

$f$  has a maximum at  $x=1$

$f$  has no local minimum.

- (c) Find the intervals on which  $f(x)$  is concave up and concave down.

$f''(x) = 0$  when  $x=2$

$f''$  is never undefined

---- 0 +++ ← Sign of  $f''$

← 0 2 4 →

$f''(0) = \frac{-}{+} < 0$ ,  $f''(4) = \frac{+}{+} > 0$

Answer:

$f(x)$  is concave up on  $(2, \infty)$  and concave down on  $(-\infty, 2)$ .

- (d) Find the  $x$ -values of any inflection points of  $f(x)$ . If there aren't any, you must explicitly state this and justify your answer.

$f$  has an inflection point at  $x=2$ .

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Note the function and its derivatives are:

$$f(x) = \frac{x}{e^x}, \quad f'(x) = \frac{-(x-1)}{e^x}, \quad f''(x) = \frac{x-2}{e^x}.$$

- (e) Give the equation of any horizontal asymptotes of  $f(x)$  or state that none exist. Justify your answer using Calculus.

For horizontal asymptotes, check limit of  $f(x)$

as  $x \rightarrow \pm\infty$ .

$$\lim_{x \rightarrow +\infty} \frac{x}{e^x} \stackrel{④}{=} \lim_{x \rightarrow \infty} \frac{1}{e^x} = 0$$

$\uparrow$  form  $\frac{0}{0}$ .

So  $y=0$  is a horizontal asymptote

$$\lim_{x \rightarrow -\infty} \frac{x}{e^x} = \lim_{x \rightarrow \infty} -xe^x = -\infty. \quad \text{So } y=0 \text{ is the } \underline{\text{only}} \text{ horizontal asymptote.}$$

- (f) Give the equation of any vertical asymptotes of  $f(x)$  or state that none exist. Justify your answer using using Calculus.

No vertical asymptotes.

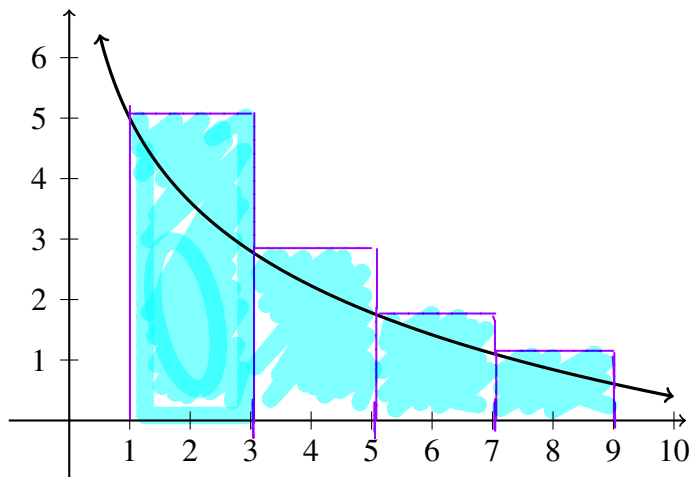
The denominator  $e^x$  is never zero.

So there is no  $x$ -value  $a$  so that

$$\lim_{x \rightarrow a^{\pm}} \frac{x}{e^x} = \pm\infty$$

6. (10 points) The function  $f(x) = 5 - 2\ln(x)$  is graphed below. We want to estimate the area under the curve  $f(x)$  on the interval  $[1, 9]$  using  $L_4$ . (That is, we want to use 4 approximating rectangles and left-hand end points.)

(a) Sketch the four approximating rectangles on the graph.



- (b) Do a calculation to estimate the area under the curve using  $L_4$  (that is, use 4 approximating rectangles and left-hand end points) and simplify your answer. Note: You are obviously not expected to compute things like  $\ln(4)$ . It is acceptable to have numbers like this in your final answer.

$$\begin{aligned}
 A &\approx 2 \left( f(1) + f(3) + f(5) + f(7) \right) \\
 &= 2 \left( 5 - 2\ln(1) + 5 - 2\ln(3) + 5 - 2\ln(5) + 5 - 2\ln(7) \right) \\
 &= 40 - 4 \left( \ln(3) + \ln(5) + \ln(7) \right)
 \end{aligned}$$

This is an acceptable final answer.

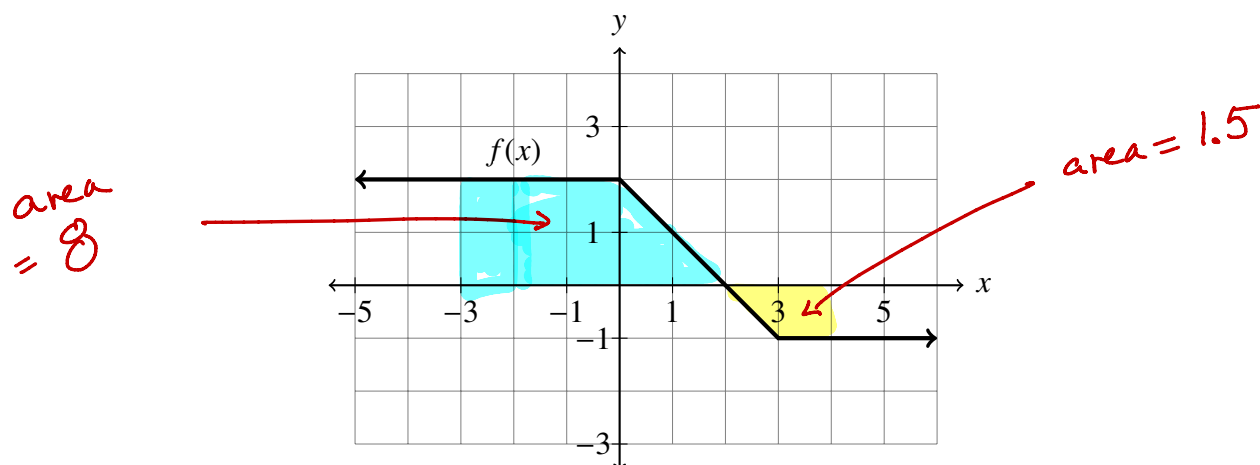
7. (10 points) Evaluate the indefinite integrals below.

$$(a) \int (5 \cos(x) + x^4 + x^{-1} + 8) dx$$

$$= 5 \sin(x) + \frac{1}{5} x^5 + \ln|x| + 8x + C$$

$$(b) \int \frac{1+x^3}{x^2} dx = \int (x^{-2} + x) dx = -x^{-1} + \frac{1}{2} x^2 + C$$

8. (10 points) Evaluate the definite integrals below using the graph of  $f(x)$  and properties of definite integrals. Show your work.



$$(a) \int_{-3}^4 f(x) dx = 8 - 1.5 = \underline{\underline{6.5}}$$

$$(b) \int_{-3}^4 (2f(x) + 5) dx = 2 \int_{-3}^4 f(x) dx + \int_{-3}^4 5 dx$$

$$= 2(\underline{\underline{6.5}}) + 5(7) = 13 + 35 = 48$$

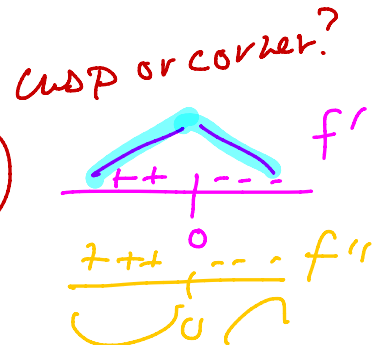
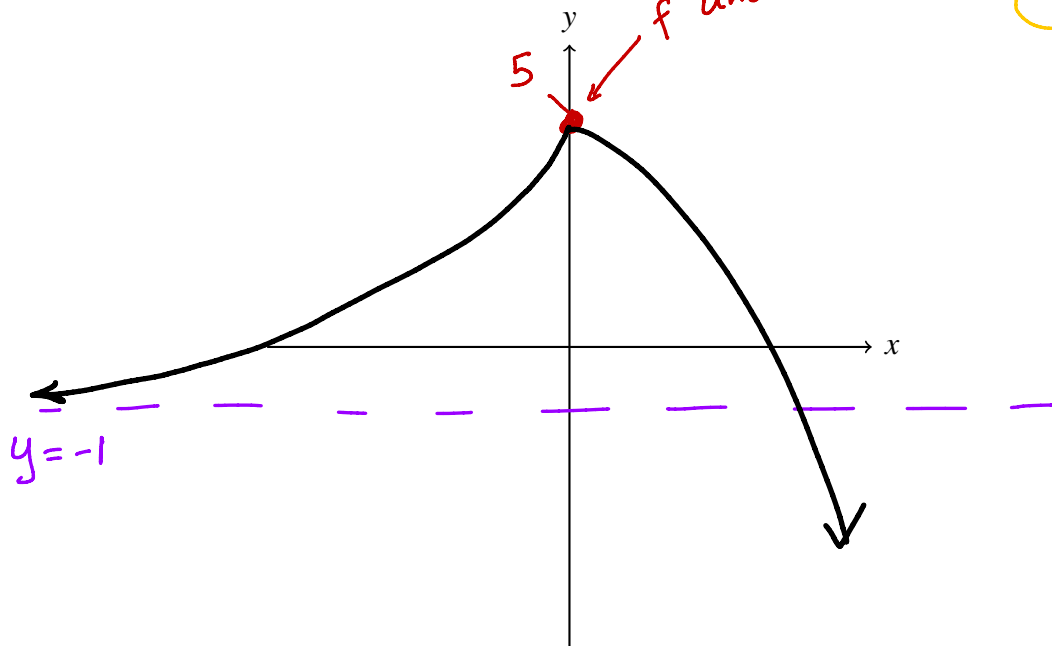
8

↑ height      ↑ base



9. (12 points) Sketch the graph of a function  $f(x)$  that satisfies all of the given conditions. Clearly label any important points on the  $x$ -axis, draw any asymptotes clearly as dashed lines, and label any asymptotes with their equations.

- (a) The domain of  $f(x)$  is  $(-\infty, \infty)$ .
- ✓ (b)  $f(0) = 5$  and  $f'(0)$  is undefined.
- ✓ (c)  $\lim_{x \rightarrow -\infty} f(x) = -1$
- (d)  $f'(x) > 0$  on the interval  $(-\infty, 0)$ ;  $f'(x) < 0$  on the interval  $(0, \infty)$
- (e)  $f''(x) > 0$  on the interval  $(-\infty, 0)$ ;  $f''(x) < 0$  on the interval  $(0, \infty)$



**Extra Credit:** Suppose  $C(t)$  models the position of a car and  $B(t)$  models the position of a bike over the same time interval,  $[0, 2]$ , where  $C$  and  $B$  are measured in miles and  $t$  in hours.

(3 pts) Translate the following sentence into the language of Calculus: "The car goes faster than the bike but the bike accelerates faster than the car." (i.e. rewrite the sentence using derivatives in some form.)

(2pts) Construct a pair of functions  $C(t)$  and  $B(t)$  satisfying the properties described in the sentence on the interval  $[0, 2]$ . (Note, your functions do not have to be realistic...)

translation:  $C'(t) > B'(t)$  but  $B''(t) > C''(t)$

reverse engineer an example: Make  $B''(t) = 5$  and  $C''(t) = 0$   
 So  $B'(t) = 5t + C$  and  $C'(t) = D$ . Pick  $C = 0$  and  $D = 100$ . ← Note: Irresponsibly high speed!

So  $B'(t) = 5t$  and  $C'(t) = 100$ .

ANSWER:  $B(t) = \frac{5}{2}t^2$ ,  $C(t) = 100t$ .