Fall 2023 **Math F251X**

Calculus I: Final Exam

Name: Solutions

Section: ☐ 9:15 (Mohamed Nouh)
☐ 11:45 (James Gossell)
☐ Online (Leah Berman)

Rules:

- Partial credit will be awarded, but you must **show your work**.
- You may have a single handwritten 3″ × 5″ notecard, both sides.
- Calculators are **not allowed**.
- Place a box around your FINAL ANSWER to each question where appropriate.
- Turn off anything that might go beep during the exam.
- You have two hours to complete the exam.

Problem	Possible	Score
1	8	
2	8	
3	10	
4	7	
5	7	
6	15	
7	10	
8	6	
9	8	
10	11	
11	10	
Extra Credit	(5)	
Total	100	

1. (8 points)

Compute the following **indefinite integrals**. Show your work. Clearly indicate any substitutions you use in such a way that someone else can follow your work.

a. $\int \left(\sin(2x) + e^x - \frac{\sqrt[3]{x}}{6} + \frac{1}{\sqrt{1-x^2}} \right) dx = \int \sin(2x)dx + \int e^x dx - \frac{1}{6}\int x^{1/3}dx + \int \frac{1}{\sqrt{1-x^2}} dx$

$u = 2x \quad \frac{du}{2} = dx$

$= \int \frac{1}{2}\sin(u)du + e^x - \frac{1}{6}\cdot\frac{x^{4/3}}{4/3} + \arcsin(x) + C = -\frac{1}{2}\cos(2x) + e^x - \frac{1}{6}\left(\frac{3}{4}\right)x^{4/3} + \arcsin(x) + C$

b. $\int \frac{\ln(2x+1)}{4x+2}\, dx = \int \frac{u}{2(2x+1)} \cdot (2x+1)\,du$

$u = \ln(2x+1)$

$du = \frac{1}{2x+1} dx$

$= \int \frac{u}{2} du = \frac{1}{2}\frac{u^2}{2} + C = \frac{1}{4}(\ln(2x+1))^2 + C$

2. (8 points)

Compute the following **limits**. Show your work clearly. Make sure you use **limit notation** where required; an answer that does not use proper notation will not receive full credit. Use = to show things are equal. If you use L'Hôpital's rule, write $\overset{H}{=}$ or $\overset{L'H}{=}$ to indicate where you are applying it.

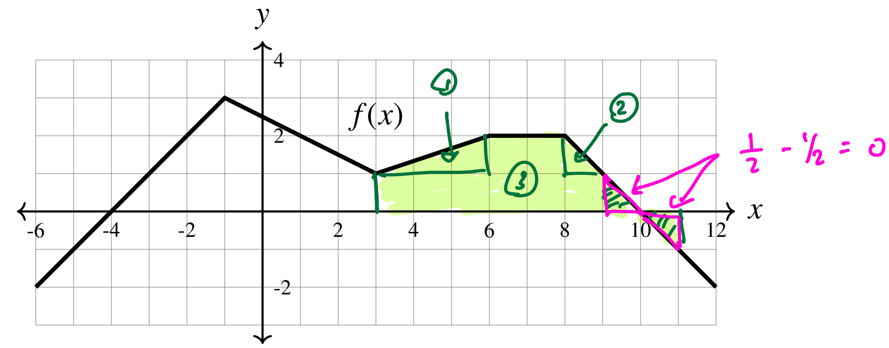
a. $\lim_{x\to\infty} \frac{6x^4 - 5x^2}{7x^4 - 14} \cdot \frac{1/x^4}{1/x^4} = \lim_{x\to\infty} \frac{6 - 5/x^2}{7 - 14/x^2} = \frac{6}{7}$

b. $\lim_{x\to 3} \frac{2x^2 - 5x - 3}{x^2 - 9}$ type $\frac{18-15-3}{9-9} = 0/0$

$= \lim_{x\to 3} \frac{(2x+1)(x-3)}{(x-3)(x+3)} = \lim_{x\to 3} \frac{2x+1}{x+3} = \frac{2(3)+1}{3+3} = \frac{7}{3}$

3. (10 points)

Consider the graph of the function $f(x)$ shown below:



a. Compute $\int_3^{11} f(x)\,dx$. = shaded area

= ① + ② + ③

$= \frac{1}{2}(3)(1) + \frac{1}{2} + 8$

$= \frac{3}{2} + \frac{1}{2} + 8 = 10$

b. Let $F(t) = \int_{-6}^{t} f(x)\,dx$. **On the interval** $[-6, 3]$, at what t-value does $F(t)$ have an absolute maximum? Where does it have an absolute minumum?

- $F(t)$ has a maximum at $t =$ 3 after $x = -2$, $F(t)$ is accumulating positive area
- $F(t)$ has a minimum at $t =$ -4 $f = F'$ goes from $-$ to $+$

c. Determine $f'(1) =$ $-\frac{1}{2}$ slope of $f(t)$ at $x = 1$

d. What can you say about $\lim_{x\to-1} f'(x)$? (Note that's asking for the limit of the derivative, not the function.) Explain your answer.

$\lim_{x\to-1} f'(x)$ DNE, since $\lim_{x\to-1^-} f'(x) = 1$ and $\lim_{x\to-1^+} f'(x) = -\frac{1}{2}$

slope of TL to f as $x \to 1^-$ and $x \to 1^+$

4. (7 points)

A portion of the implicitly defined curve $3 + xy^2 = x^3$ is shown below.

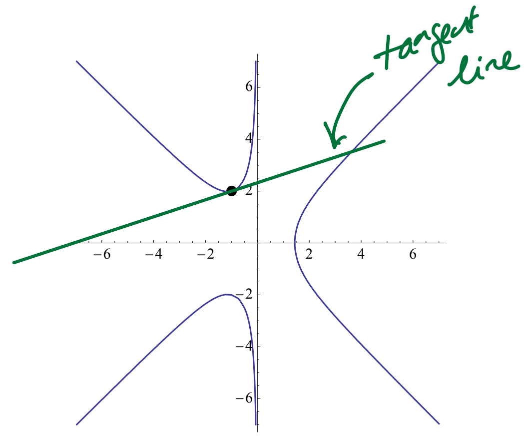
a. Determine $\frac{dy}{dx}$.

$\frac{d}{dx}\left(3 + xy^2\right) = \frac{d}{dx}\left(x^3\right) \Rightarrow$

$x\left(2y\frac{dy}{dx}\right) + y^2 = 3x^2 \Rightarrow$

$\frac{dy}{dx}(2xy) = 3x^2 - y^2 \Rightarrow$

$\frac{dy}{dx} = \frac{3x^2 - y^2}{2xy}$



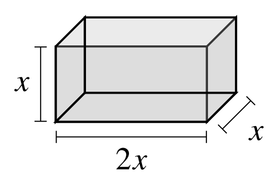
b. Write the equation of the tangent line to the curve at the point $(-1, 2)$, which is shown with a black dot on the curve. **Clearly draw and label** the tangent line on the graph.

at $(-1, 2)$, $\frac{dy}{dx} = \frac{3(-1)^2 - (2)^2}{2(-1)(2)} = \frac{3-4}{-4} = 1/4$

so $y = 1/4(x+1) + 2$ (which is at least plausible)

5. (7 points)

Suppose a rectangular box has two square sides and four sides where the length of the side is two times the width of the side (see diagram). Suppose the short side of the box is measured to be 10 cm ± 1 mm (1mm = 1/10 cm), and the volume is computed. Use linearization or differentials to determine the **error** in the measurement of the volume. Write your answer with a sentence, using correct units.



$V = (2x)(x)(x) = 2x^3$. When $x = 10$cm, $V = 2(10)^3 = 2000$ cm^3.

$\frac{\Delta V}{\Delta x} \approx \frac{dV}{dx} = 6x^2 \rightarrow \Delta V \approx 6x^2 \Delta x$ when $x = 10$, $\Delta x = \frac{1}{10}$ cm

$\Rightarrow \Delta V = 6(10)^2(\pm\frac{1}{10}) = \pm 60$ cm^3 and $\frac{\Delta V}{V} = \frac{60}{2000}$

$= \frac{30}{1000}$

$= \frac{3}{100}$

The error in the volume is ± 60cm^3.

The relative error in the measurement of the volume is $\pm\frac{3}{100} = \pm 0.03$cm^3.

6. (15 points)

A hot air balloon rises with an upward velocity of $v(t) = 2te^{-t^2} = \dfrac{2t}{e^{t^2}}$ kilometers per minute (km/min), t minutes after it is launched ($t \geq 0$).

a. What is the balloon's **initial acceleration**, $a(0)$? Include units in your answer.

$$a(t) = v'(t) = 2t(e^{-t^2})(-2t) + e^{-t^2}(2) = -4t^2e^{-t^2} + 2e^{-t^2}$$
$$= (2e^{-t^2})(1-2t^2)$$

$a(0) = 2(1) = 2$. The initial acceleration is 2 km/min^2

b. At what **time** does the balloon reach its **maximum upward velocity**? Use calculus techniques to **verify** that the time you find really is where the velocity is a maximum. Include units in your answer, and show your work.

$$v'(t) = 0 \Rightarrow \frac{1-2t^2}{2e^{t^2}} = 0 \Rightarrow 1-2t^2 = 0 \Rightarrow t^2 = \frac{1}{2} \Rightarrow t = \pm\frac{1}{\sqrt{2}}$$

take positive value.

Is it a max? $a'(t) = v''(t) = (2e^{-t^2})(-4t) + (1-2t^2)(2e^{-t^2})(-2t)$
$$= (-2t)(2e^{-t^2})(+2 + (1-2t^2))$$

$$a'\left(\tfrac{1}{\sqrt{2}}\right) = \left(-2\cdot\tfrac{1}{\sqrt{2}}\right)(2e^{-1/2})\left(+2+1-2\left(\tfrac{1}{2}\right)\right) = (-)(+)(+) = -$$

so $v(t)$ is concave down at $t = 1/\sqrt{2}$ and so $t = 1/\sqrt{2}$ is a maximum.

The balloon reaches its max velocity at $t = 1/\sqrt{2}$ seconds after launch.

c. Evaluate the integral $\displaystyle\int_0^2 v(t)\,dt$. Show your work and simplify your answer as much as possible.

$$\int_0^2 \frac{2t}{e^{t^2}}\,dt = \int_0^2 2te^{-t^2}\,dt = \int_0^{-4} -e^u\,du = \int_{-4}^0 e^u\,du = e^0 - e^{-4} = 1 - \frac{1}{e^4}$$

$u = -t^2$ $\quad t=0 \Rightarrow u=0$

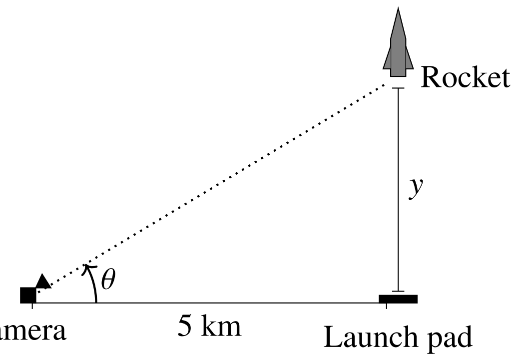
$du = -2t\,dt$ $\quad t=2 \Rightarrow u=-4$

d. Write a sentence explaining the meaning of your answer to the previous part in the context of the problem, in a way someone who has not taken calculus can understand. Include units in your answer.

During the 1st 2 minutes after launch, the balloon rose $1 - \frac{1}{e^4}$ kilometers.

7. (10 points)

A rocket is launched vertically off of a launch pad. A camera is positioned 5 kilometers from the launch pad. When the rocket is 12 kilometers above the launch pad, its velocity is 2 km/sec. (See the diagram below.)



When $y = 12$, know $\frac{dy}{dt} = 2$ km/s.

We want to find $\frac{d\theta}{dt}$.

Find the necessary **rate of change** of the camera's angle θ so that it stays focused on the rocket at the instant when the rocket is 12 kilometers above the launch pad. Answer the question with a **sentence**, including correct units.

Know $\frac{y}{5} = \tan\theta \Rightarrow y = 5\tan\theta$ or $\theta = \arctan\left(\frac{y}{5}\right)$.

If $\theta = \arctan\left(\frac{y}{5}\right)$:

Then $\frac{d\theta}{dt} = \left(\frac{1}{1+\left(\frac{y}{5}\right)^2}\right)\left(\frac{1}{5}\right)\frac{dy}{dt}$

$= \left(\frac{1}{1+\left(\frac{12}{5}\right)^2}\right)\left(\frac{1}{5}\right)(2)$

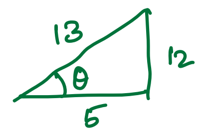
$= \left(\frac{5^2}{5^2+12^2}\right)\left(\frac{2}{5}\right)$

$= \frac{10}{13^2}$

The angle is changing at a rate of $\frac{10}{13^2}$ rad/s.

If $y = 5\tan\theta$:

$\frac{dy}{dt} = 5\sec^2\theta\,\frac{d\theta}{dt} = \frac{5}{\cos^2\theta}\frac{d\theta}{dt}$



$\Rightarrow \cos^2\theta = \left(\frac{5}{13}\right)^2$

$\Rightarrow \sec^2\theta = \frac{13^2}{5^2}$

So $2 = 5 \cdot \frac{13^2}{5^2}\frac{d\theta}{dt} \Rightarrow$

$\frac{d\theta}{dt} = \frac{2\cdot 5^2}{5\cdot 13^2} = \frac{10}{13^2}$

In this version also, the angle is changing at a rate of $\frac{10}{13^2}$ rad/s.

8. (6 points)

The population of rabbits in a local park, measured since 2011, can be modeled by the equation

$$P(t) = \frac{10\,000e^{t/10}}{480 + 20e^{t/10}}$$

where t measures time, in years, since 2011.

a. How many rabbits were in the park in 2011? Simplify your answer.

$P(0) = \dfrac{1000(1)}{480+20(1)} = \dfrac{1000}{500} = 200.$ The park started with 200 rabbits.

b. Compute $\lim_{t\to\infty} P(t)$. Show your work clearly, with correct use of notation.

$\lim_{t\to\infty} P(t) \overset{L'H}{=} \lim_{t\to\infty} \dfrac{10000(e^{t/10})(1/10)}{(20e^{t/10})(1/10)} = \dfrac{10000}{20} = \dfrac{1000}{2} = 500$

type $\frac{\infty}{\infty}$

c. Write a sentence that explains the meaning of the limit you just calculated, in terms a person who has not taken calculus can understand.

Eventually, the population of rabbits will stabilize at 500 rabbits.

9. (8 points)

Compute the following **derivatives**. Show your work. You do NOT need to simplify your answer.

a. $f(x) = \sqrt{x}\left(\ln(x^3 - x^2)\right)$ $= x^{1/2}(\ln(x^3 - x^2))$

$f'(x) = x^{1/2}\left(\frac{1}{x^3 - x^2}\right)(3x^2 - 2x) + \ln(x^3 - x^2)\left(\frac{1}{2}x^{-1/2}\right)$

b. $g(x) = \left(e^{-x} + \dfrac{\arctan(x)}{2}\right)^5$. (Note $\arctan(x) = \tan^{-1}(x)$.)

$g'(x) = 5\left(e^{-x} + \frac{1}{2}\arctan(x)\right)^4\left(e^{-x}(-1) + \frac{1}{2}\left(\frac{1}{1+x^2}\right)\right)$

10. (11 points)

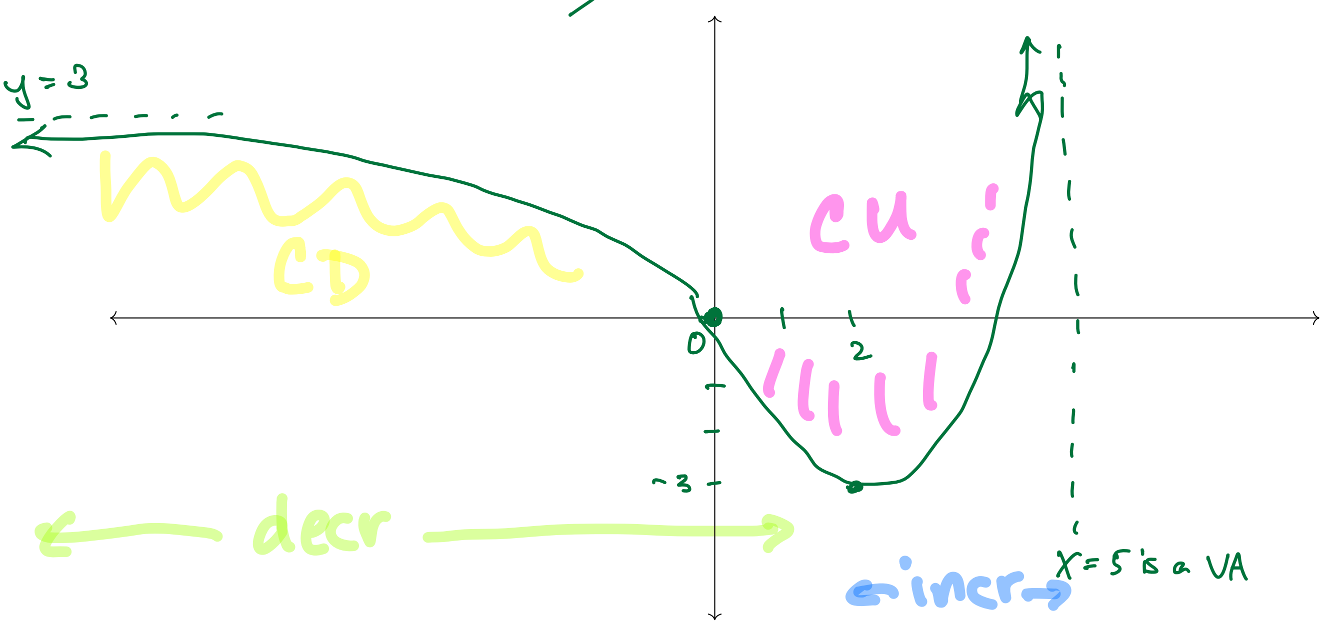
Sketch a graph of a function $h(x)$ that satisfies all of the following properties.

After drawing the graph:

- **Label** on the graph the following things, if they exist, by drawing a point on the graph and labeling: any local maximums by writing LOCAL MAX, local minimums by writing LOCAL MIN, inflection points by writing IP
- **Draw** any horizontal and vertical asymptotes with dashed lines and **label** them with their equation.
- **Mark** any important x-values and y-values (with numbers) on the x- and y-axes.

Properties:

- The domain of $h(x)$ is $(-\infty, 5)$
- $h(0) = 0$ and $h(2) = -3$
- $h'(x) < 0$ on the interval $(-\infty, 2)$
- $h'(x) > 0$ on the interval $(2, 5)$
- $h''(x) < 0$ on the interval $(-\infty, 0)$ CD on $(-\infty, 0)$
- $h''(x) > 0$ on the interval $(0, 5)$ CU on $(0, 5)$
- $\lim_{x\to-\infty} h(x) = 3$ $y = 3$ is HA to $\leftarrow$
- $\lim_{x\to5^-} h(x) = +\infty$ $x = 5$ is VA



11. (10 points)

A hot cup of coffee in a room whose ambient temperature is 68°F is changing temperature at a rate of $R(t)$, where t is measured in minutes and $R(t)$ is measured in °F/minute.

a. Write down a complete sentence carefully explaining the meaning of $R(5) = -9$ in the context of the problem. Use units in your answer.

At the instant 5 minutes has passed, the coffee is cooling at a rate of 9 °F/min

b. Would you expect $R(t) > 0$ or $R(t) < 0$? Explain your answer in a sentence, given the context of the problem.

We expect $R(t) < 0$ since the coffee is cooling down

c. Write a complete sentence explaining the meaning of the quantity

$$\int_0^8 R(t)\,dt = -107$$

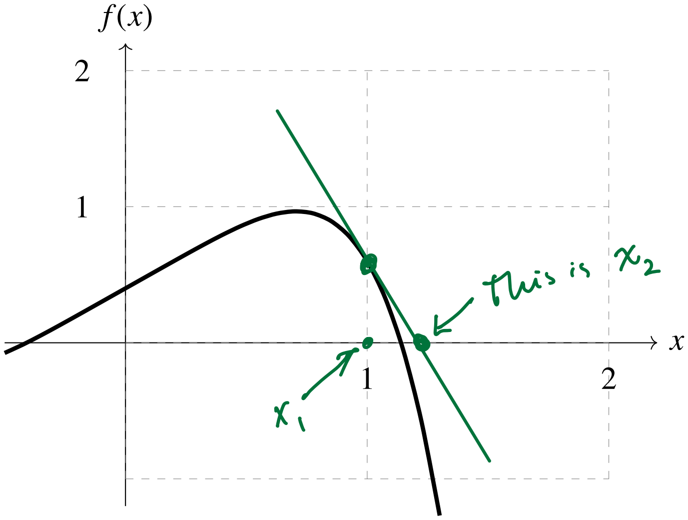
in the context of the problem. Include units in your answer.

During the first 8 minutes, the coffee cooled by 107 °F

d. Assume the cup of coffee started at 200°F and was left sitting on the kitchen counter, untouched. Write an expression that you would need to compute to determine the temperature of the coffee an hour later.

Temp after 1 hour $= 200 + \int_0^{60} R(t)\,dt$.

Extra Credit (5 points) A portion of the graph of the function $f(x) = -\frac{4}{5}x^5 + x + \frac{2}{5}$ is shown below.



a. Suppose Newton's method is used to find an approximate solution to $f(x) = 0$ from an initial guess of $x_1 = 1$. **Sketch** on the graph how the next approximation x_2 will be found, **labeling** its location on the x-axis.

b. If your starting guess is $x_1 = 1$, **compute** x_2.

$f(x) = -\frac{4}{5}x^5 + x + 2/5$

$f(1) = -\frac{4}{5} + 1 + \frac{2}{5} = 1 - \frac{2}{5} = \frac{3}{5}$

$f'(x) = -\frac{4}{5}(5x^4) + 1 = -4x^4 + 1$ so $f'(1) = -4 + 1 = -3$ (plausible!)

So TL: $y = -3(x-1) + 3/5$. TL $= 0 \Rightarrow 0 = -3(x-1) + 3/5 \Rightarrow \frac{-3}{5} = -3(x-1)$

$\Rightarrow \frac{1}{5} = x - 1$

$\Rightarrow \boxed{x = 6/5 = x_2}$

Alternately, use the formula: TL to $(x_1, f(x_1))$

is $y = f'(x_1)(x - x_1) + f(x_1) = 0 \Rightarrow$

$x - x_1 = -\frac{f(x_1)}{f'(x_1)} \Rightarrow x = x_1 - \frac{f(x_1)}{f'(x_1)} = x_2$

So $x_2 = 1 - \frac{3/5}{-3} = 1 + \frac{1}{5} = 6/5$.