Fall 2023 **Math F251X**

Calculus I: Midterm 2

Name: Solutions

Rules:

- Partial credit will be awarded, but you must **show your work**.
- You may have a single handwritten $3'' \times 5''$ notecard, both sides.
- Calculators are **not allowed**.
- Place a box around your $\boxed{\text{FINAL ANSWER}}$ to each question where appropriate.
- Turn off anything that might go beep during the exam.

Good luck!

Problem	Possible	Score
1	8	
2	7	
3	18	
4	11	
5	12	
6	12	
7	10	
8	12	
9	10	
Extra Credit	5	
Total	100	

1. (8 points)

Consider the function $f(x) = \sqrt[3]{x}$.

a. Determine the **linear approximation** $L(x)$ of the function at the point $a = 64 = 4^3$.

$f'(x) = \frac{d}{dx}(x^{1/3}) = \frac{1}{3}x^{-2/3} = \frac{1}{3(\sqrt[3]{x})^2}$

$\begin{array}{r} 16 \\ \times\ 3 \\ \hline 48 \end{array}$

$f'(64) = \frac{1}{3(4)^2} = \frac{1}{3 \cdot 16} = \frac{1}{48}$

$f(64) = 4$

So $L(x) = \frac{1}{48}(x-64) + 4$

b. Use the linear approximation you just found to **estimate** $\sqrt[3]{70}$.

$\sqrt[3]{70} \approx L(70) = \frac{1}{48}(70-64) + 4$

$= \frac{1}{48}(6) + 4 = \frac{1}{8} + 4$

(Note this makes sense: $\sqrt[3]{70}$ should be a little bit bigger than 4, which it is!)

2. (7 points)

Consider the function

$$j(t) = \cos(t) + (\sin(t))^2.$$

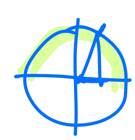
Determine the **absolute minimum** and **absolute maximum** values of $j(t)$ on the interval $[0, \pi]$. Show your work.

$j'(t) = -\sin(t) + 2\sin(t)\cos(t) = \sin(t)(-1 + 2\cos(t))$

$j'(t) = 0 \Rightarrow \sin(t) = 0$ or $-1 + 2\cos t = 0 \Rightarrow \cos(t) = 1/2$

$\Rightarrow t = 0$ or $t = \pi/3$

$t = \pi$



$j(0) = \cos(0) + (\sin(0))^2 = 1$

$j(\pi) = \cos(\pi) + (\sin(\pi))^2 = -1$

$j(\pi/3) = \cos(\pi/3) + \sin(\pi/3)^2 = 1/2 + \left(\frac{\sqrt{3}}{2}\right)^2$

$= \frac{1}{2} + \frac{3}{4} = \frac{2}{4} + \frac{3}{4} = \frac{5}{4}$

Absolute maximum: $5/4$

Absolute minimum: -1

3. (18 points)

Answer the questions below about the function $f(x) = \dfrac{x^4}{e^x}$. It is a fact that after simplification,

$$f'(x) = \frac{-x^3(x-4)}{e^x}, \qquad \text{and } f''(x) = \frac{x^2(x-6)(x-2)}{e^x}$$

You must show your work and justify your conclusion with a few words or a computation. Make sure someone else can follow your work.

a. Determine the intervals where f is **increasing** and where f is **decreasing**. Show your work.

$f'(x) = 0 \Rightarrow x = 0$ or $x = 4$

$f'(x)$ never DNE, since $e^x > 0$ for all x

x	-1	0	1	4	5
f'	$-$	0	$+$	0	$-$
f	$\searrow$	MIN	$\nearrow$	MAX	$\searrow$

$f'(-1) = \dfrac{-(-1)^3(-5)}{+} = (-)(-)(-) = -$

$f'(1) = \dfrac{-(1)^3(1-4)}{+} = (-)(-) = +$

$f'(5) = \dfrac{-(5)^3(5-4)}{+} = (-)(+)(+) = -$

Increasing: $(0, 4)$ Decreasing: $(-\infty, 0) \cup (4, \infty)$ (If none write "none".)

b. Fill in the blanks: $f(x)$ has a local maximum at $x =$ 4 and a local minimum at $x =$ 0. (If none, write "none".)

c. Find all intervals where f is **concave up** and where f is **concave down**. Show your work.

$f''(x) = 0 \Rightarrow x = 0$ or $x = 6$ or $x = 2$

$f''(x)$ never DNE

x	-1	0	1	2	3	6	7
f''	$+$		$+$		$-$		$+$
f	$\cup$		$\cup$	IP	$\cap$	IP	$\cup$

$f''(-1) = +(-)(-) = +$

$f''(1) = +(-)(-) = +$

$f''(3) = +(-)(+) = -$

$f''(7) = +(+)(+) = +$

Concave up: $(-\infty, 2) \cup (6, \infty)$ Concave down: $(2, 6)$ (If none write "none".)

d. Fill in the blanks: $f(x)$ has (an) inflection point(s) at $x =$ $2, 6$. (If none, write "none".)

4. (11 points)

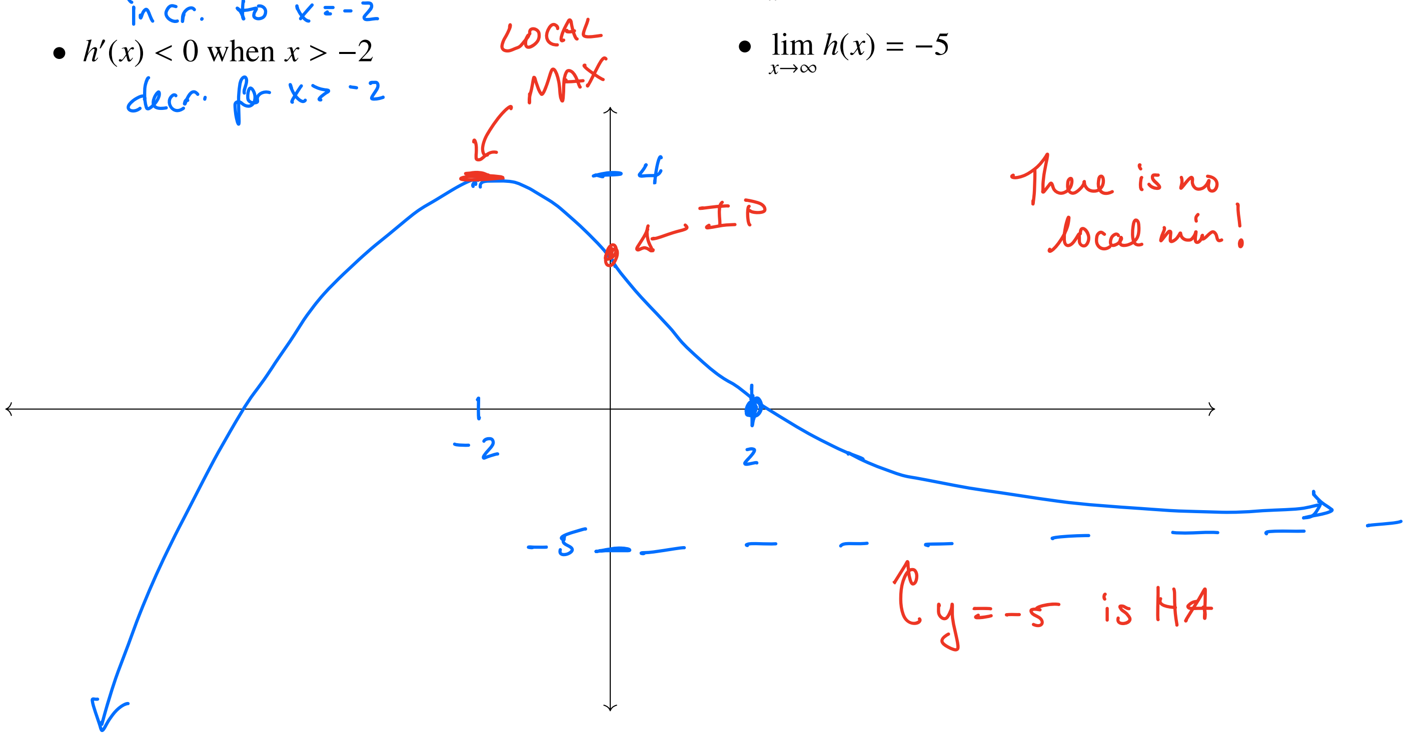
Sketch a graph of a function $h(x)$ that satisfies all of the following properties.

After drawing the graph:

- **Label** on the graph the following things, if they exist, by drawing a point on the graph and labeling: any local maximums by writing LOCAL MAX, local minimums by writing LOCAL MIN, inflection points by writing IP
- Draw any horizontal and vertical asymptotes with dashed lines and **label** them with their equation.
- Mark any important x-values and y-values on the x- and y-axes.

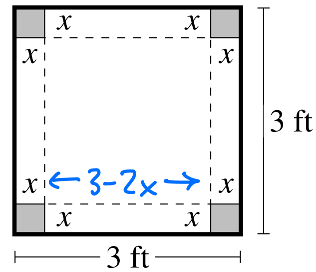
Properties:

- The domain of $h(x)$ is $(-\infty, \infty)$
- $h(-2) = 4$ and $h(2) = 0$
- $h'(x) > 0$ when $x < -2$
  incr. to $x=-2$
- $h'(x) < 0$ when $x > -2$
  decr. for $x > -2$
- $h''(x) < 0$ when $x < 0$ CD to 0
- $h''(x) > 0$ when $x > 0$ Cu after 0
- $\lim_{x\to-\infty} h(x) = -\infty$
- $\lim_{x\to\infty} h(x) = -5$



5. (12 points)

An open box is to be constructed by cutting squares out of the four corners of a 3 foot by 3 foot piece of cardboard and folding up the sides. (See the diagram. Note that the box will not have a lid, and the height of the box will be x feet.)



a. Write an equation for the **volume** of the box in terms of the variable x.

$$V(x) = x(3-2x)(3-2x)$$
$$= (3x-2x^2)(3-2x)$$
$$= 9x - 6x^2 - 6x^2 + 4x^3 = 4x^3 - 12x^2 + 9x$$

b. Determine the **dimensions** of the box with the largest volume. Show your work, and use calculus to **justify** that your answer is the maximum. Include units in your final answer. An answer with no clear justification will not receive full credit.

$$V'(x) = 12x^2 - 24x + 9 = 3(4x^2 - 8x + 3) = 12x^2 - 24x + 9$$
$$= 3(2x-1)(2x-3)$$

$V'(x) = 0 \Rightarrow x = 1/2$ or $x = 3/2$

Note domain is $[0, 3/2]$

$V(0) = 0$

$V(3/2) = 0$

$V(1/2) = \frac{1}{2}(3 - 2\cdot 1/2)^2$

$= \frac{1}{2}(2)^2 = 4$

So $x = 1/2$ is a ABS max!

Check with 2^{nd} deriv test:

$V''(x) = 24x - 24$

$V''(1/2) = 12 - 24 < 0$

$\Rightarrow 1/2$ is, indeed, $\cap$

$\Rightarrow$ max

when $x = 1/2$, $3 - 2x = 3 - 2(1/2) = 3 - 1 = 2$

Dimensions: length: 2 width: 2 height: 1/2

6. (12 points)

Evaluate the following limits. Show your work. If you use L'Hôpital's rule, you must indicate where you are using it by writing $\overset{H}{=}$ or $\overset{L'H}{=}$ or something similar.

a. $\lim_{x\to 2} \frac{3^x - x^3 - 1}{x - 2}$ type $\frac{3^2 - 2^3 - 1}{2-2} = \frac{0}{0}$

$\overset{L'H}{=} \lim_{x\to 2} \frac{3^x \ln(3) - 3x^2}{1}$

$= 3^2 \ln(3) - 3(4) = \boxed{9\ln(3) - 12}$

b. $\lim_{x\to 0} \frac{x^2 - x}{2\sin x - 2}$ type $\frac{0-0}{2(0)-2}$

$= \frac{0}{-2}$

$= 0$ (No L'H needed! We can just evaluate this limit by direct substitution!)

c. $\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}$ type $\frac{\infty}{\infty}$

$= \lim_{x\to\infty} \frac{\ln(x)}{x^{1/2}}$

$\overset{L'H}{=} \lim_{x\to\infty} \frac{1/x}{\frac{1}{2}x^{-1/2}} = \lim_{x\to\infty} \frac{1}{2x} \cdot x^{1/2} = \lim_{x\to\infty} \frac{1}{2x^{1/2}} = 0$

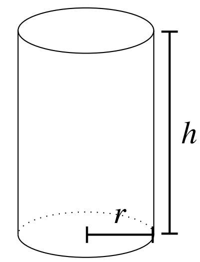
d. $\lim_{x\to\infty} xe^{-x}$ type $\infty \cdot 0$

$= \lim_{x\to\infty} \frac{x}{e^x}$ type $\frac{\infty}{\infty}$

$\overset{L'H}{=} \lim_{x\to\infty} \frac{1}{e^x} = 0$

7. (10 points)

A giant ice cylinder, being prepared for the World Ice Art Championships, is carved to have a radius of 100 cm and a height of 300 cm. The cylinder is uniformly melting, which means that the ratio of the radius to the height remains constant: specifically, $\frac{r}{h} = \frac{1}{3}$. If the height is **decreasing** at a rate of 2 cm/hour, how fast is the surface area changing when the cylinder is 150 cm tall?



Write your answer in a complete sentence, using units.

Recall that the surface area of a cylinder with radius r and height h is $S = 2\pi rh + 2\pi r^2$.

Know: $\frac{dh}{dt} = 2$ cm/h and $r = \frac{1}{3}h$

Want $\frac{dS}{dt}$ when $h = 150$ cm

$$S = 2\pi\left(\frac{h}{3}\right)\cdot h + 2\pi\left(\frac{h}{3}\right)^2$$

$$= \frac{2\pi}{3}h^2 + \frac{2\pi}{9}h^2 = \frac{8\pi}{9}h^2$$

$$\begin{array}{r} 32 \\ 50 \\ \hline 1600 \end{array}$$

$$\frac{dS}{dt} = \frac{16\pi}{9}h\frac{dh}{dt}$$

When $h = 150$ & $\frac{dh}{dt} = -2$, $\frac{dS}{dt} = \frac{16\pi}{9}(150)(-2) = -\frac{32\pi(50)}{3}$

The surface area is decreasing at a rate of $\frac{1600\pi}{3}\ \frac{cm^2}{hour}$.

The hard way: $\frac{dS}{dt} = 2\pi\left(\frac{dr}{dt}\cdot h + \frac{dh}{dt}\cdot r\right) + 4\pi r\frac{dr}{dt}$

and $r = \frac{1}{3}h \Rightarrow \frac{dr}{dt} = \frac{1}{3}\frac{dh}{dt}$ and when $h = 150$, $r = \frac{150}{3} = 50$, when $\frac{dh}{dt} = -2$

$\frac{dr}{dt} = \frac{-2}{3}$

so $\frac{dS}{dt} = 2\pi\left(-\frac{2}{3}\cdot 150 + -2(50)\right) + 4\pi(50)\left(\frac{-2}{3}\right)$

$$= 2\pi(-100 + (-100)) + -\frac{400\pi}{3}$$

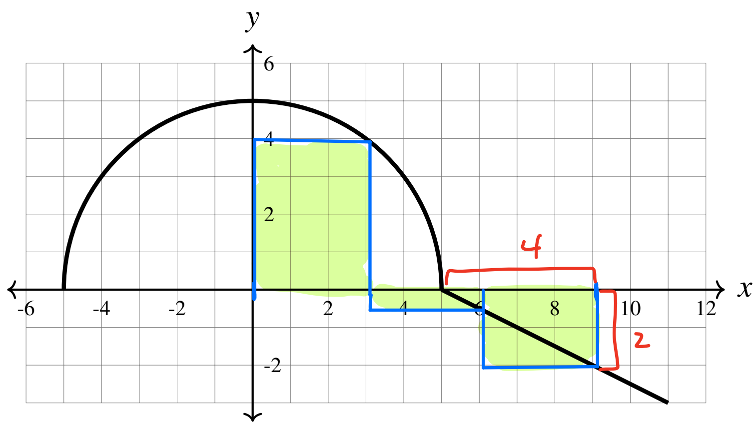
$$= -400\pi - \frac{400\pi}{3} = -\frac{1200\pi}{3} - \frac{400\pi}{3} = \frac{-1600\pi}{3}$$

8. (12 points)

A portion of the function

$$f(x) = \begin{cases} \sqrt{25-x^2} & -5 \le x \le 5 \\ -\frac{1}{2}(x-5) & x \ge 5 \end{cases}$$

is graphed below.



a. We want to approximate $\int_0^9 f(x)\,dx$ using **three** right-hand rectangles.

i. **Draw** the three right-hand rectangles on the graph. Lightly shade them in. Make sure I can tell where your rectangles are and try to be reasonably precise.

ii. Now **approximate** $\int_0^9 f(x)\,dx$ using the three right-hand rectangles. (Your answer should be a number.)

Note $f(6) =$
$-\frac{1}{2}(6-5) = -1/2$

area $\approx 3(4) + 3(-1/2) + 3(-2)$
$= 3(2 - 1/2)$
$= 3\left(\frac{3}{2}\right) = \boxed{\frac{9}{2}}$

b. Use geometry to compute $\int_0^9 f(x)\,dx$ exactly. Show your work.

This is negative because it's below the x-axis!

area = (area of 1/4 circle) + (area of Δ)
$= \frac{1}{4}(\pi(5)^2) - \frac{1}{2}(4)(2) = \boxed{\frac{25\pi}{4} - 4}$

9. (10 points)

a. Determine $H(\theta) = \int \theta^{2/3} + (\sec(\theta))^2 + 5\, d\theta$. (Give the most generic answer.)

$$H(\theta) = \frac{\theta^{5/3}}{5/3} + \tan\theta + 5\theta + c$$

b. Determine $G(x) = \int \frac{14x^3 + 2x + 1}{x^3}\, dx$. (Give the most generic answer.)

$$G(x) = \int 14 + \frac{2}{x^2} + \frac{1}{x^3}\, dx = \int 14 + 2x^{-2} + x^{-3}\, dx$$

$$= 14x + \frac{2x^{-1}}{-1} + \frac{x^{-2}}{-2} + c$$

$$= 14x - \frac{2}{x} - \frac{1}{2x^2} + c$$

c. Find two different antiderivatives for the function $f(t) = 4e^t + \frac{1}{t}$.

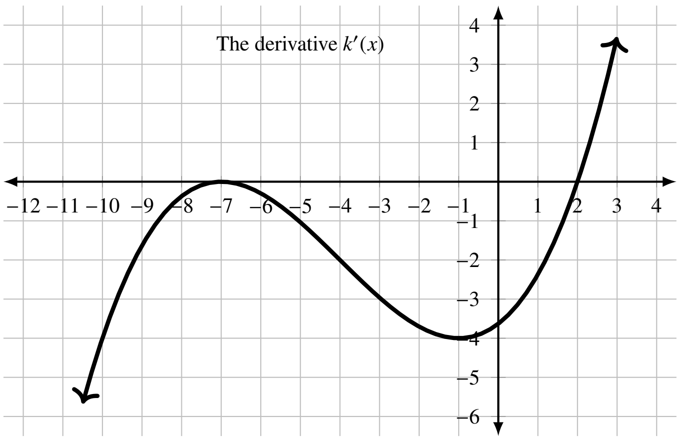
$$F(t) = 4e^t + \ln|t| + c$$

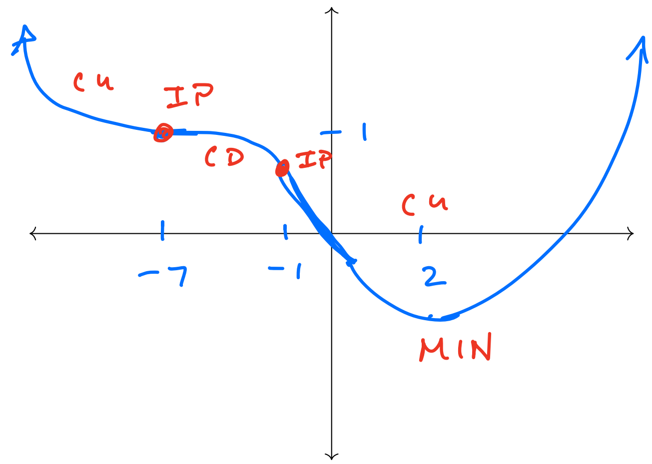
So 2 different antiderivatives are

$$F_1(t) = 4e^t + \ln|t| + 12$$

$$F_2(t) = 4e^t + \ln|t|$$

$\boxed{\text{Extra Credit}}$ (5 points) The left-hand graph shows the $\boxed{\text{DERIVATIVE}}$ k' of some function k.





The following questions are about the original function $k(x)$, <u>not</u> the graphed $k'(x)$.

a. At what value(s) of x does k have a local maximum or minimum? If none, say so. **Write some words** to explain how you know that these x-values correspond to a local max or min (or not).

Local Max: x = none Local Min: x = 2

at $x = 2$, $f'(x)$ goes negative to positive $\Rightarrow$ ↘ ↗

There is no place where f' goes positive → negative

b. At what values of x does k have inflection points? If none, say so. **Write some words** to explain how you know that these x-values correspond to inflection points (or not).

Inflection points: x = −7 and $x = -1$

at $x = -7$, the derivative of f' (that is, f'') goes + to − and at $x = -1$, f'' goes − to +

c. Sketch the graph of the original function $k(x)$ on the axes to the left of the derivative, using the fact that $k(-7) = 1$. Your sketch does not have to be to scale, but it should show the right information regarding increasing/decreasing/max/min/concavity/inflection points. Mark important values on the $x-$ and y-axes.