Fall 2023 Math F251X

Calculus I: Midterm 2

Name: ____________________

Rules:

- Partial credit will be awarded, but you must **show your work**.
- You may have a single handwritten $3'' \times 5''$ notecard, both sides.
- Calculators are **not allowed**.
- Place a box around your $\boxed{\text{FINAL ANSWER}}$ to each question where appropriate.
- Turn off anything that might go beep during the exam.

Good luck!

Problem	Possible	Score
1	8	
2	7	
3	18	
4	11	
5	12	
6	12	
7	10	
8	12	
9	10	
Extra Credit	5	
Total	100	

1. (8 points)

Consider the function $f(x) = \sqrt[3]{x}$.

a. Determine the **linear approximation** $L(x)$ of the function at the point $a = 64 = 4^3$.

b. Use the linear approximation you just found to **estimate** $\sqrt[3]{70}$.

2. (7 points)

Consider the function

$$j(t) = \cos(t) + (\sin(t))^2.$$

Determine the **absolute minimum** and **absolute maximum** values of $j(t)$ on the interval $[0, \pi]$. Show your work.

Absolute maximum: ____________

Absolute minimum: ____________

3. (18 points)

Answer the questions below about the function $f(x) = \dfrac{x^4}{e^x}$. It is a fact that after simplification,

$$f'(x) = \frac{-x^3(x-4)}{e^x}, \quad \text{and } f''(x) = \frac{x^2(x-6)(x-2)}{e^x}$$

You must show your work and justify your conclusion with a few words or a computation. Make sure someone else can follow your work.

a. Determine the intervals where f is **increasing** and where f is **decreasing**. Show your work.

Increasing: ________________ Decreasing: ________________ (If none write "none".)

b. Fill in the blanks: $f(x)$ has a local maximum at $x =$ __________ and a local minimum at $x =$ __________. (If none, write "none".)

c. Find all intervals where f is **concave up** and where f is **concave down**. Show your work.

Concave up: ________________ Concave down: ________________ (If none write "none".)

d. Fill in the blanks: $f(x)$ has (an) inflection point(s) at $x =$ __________. (If none, write "none".)

4. (11 points)

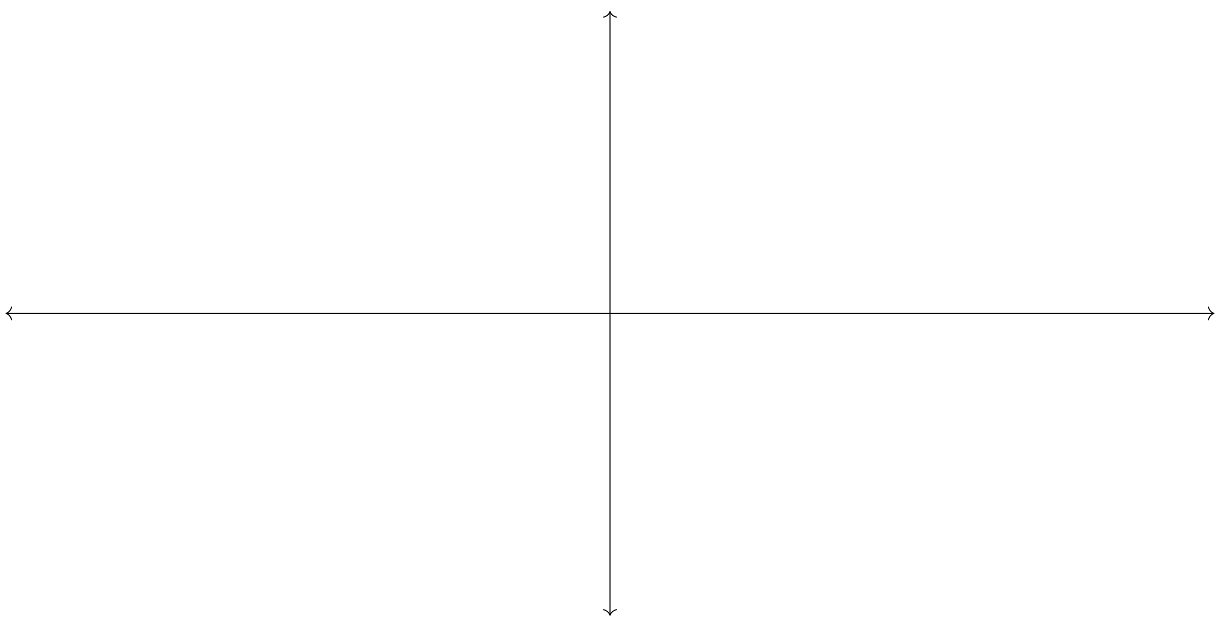
Sketch a graph of a function $h(x)$ that satisfies all of the following properties.

After drawing the graph:

- **Label** on the graph the following things, if they exist, by drawing a point on the graph and labeling: any local maximums by writing LOCAL MAX, local minimums by writing LOCAL MIN, inflection points by writing IP
- Draw any horizontal and vertical asymptotes with dashed lines and **label** them with their equation.
- Mark any important x-values and y-values on the x- and y-axes.

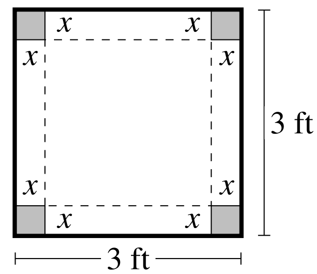
Properties:

- The domain of $h(x)$ is $(-\infty, \infty)$
- $h(-2) = 4$ and $h(2) = 0$
- $h'(x) > 0$ when $x < -2$
- $h'(x) < 0$ when $x > -2$
- $h''(x) < 0$ when $x < 0$
- $h''(x) > 0$ when $x > 0$
- $\lim_{x\to-\infty} h(x) = -\infty$
- $\lim_{x\to\infty} h(x) = -5$



5. (12 points)

An open box is to be constructed by cutting squares out of the four corners of a 3 foot by 3 foot piece of cardboard and folding up the sides. (See the diagram. Note that the box will not have a lid, and the height of the box will be x feet.)



a. Write an equation for the **volume** of the box in terms of the variable x.

b. Determine the **dimensions** of the box with the largest volume. Show your work, and use calculus to **justify** that your answer is the maximum. Include units in your final answer. An answer with no clear justification will not receive full credit.

Dimensions: length: __________ width: __________ height: __________

6. (12 points)

Evaluate the following limits. Show your work. If you use L'Hôpital's rule, you must indicate where you are using it by writing $\stackrel{H}{=}$ or $\stackrel{L'H}{=}$ or something similar.

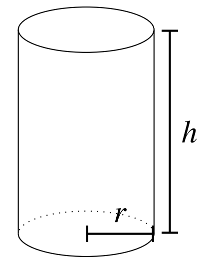
a. $\displaystyle\lim_{x\to 2} \frac{3^x - x^3 - 1}{x - 2}$

b. $\displaystyle\lim_{x\to 0} \frac{x^2 - x}{2\sin x - 2}$

c. $\displaystyle\lim_{x\to \infty} \frac{\ln x}{\sqrt{x}}$

d. $\displaystyle\lim_{x\to \infty} xe^{-x}$

7. (10 points)



A giant ice cylinder, being prepared for the World Ice Art Championships, is carved to have a radius of 100 cm and a height of 300 cm. The cylinder is uniformly melting, which means that the ratio of the radius to the height remains constant: specifically, $\frac{r}{h} = \frac{1}{3}$. If the height is **decreasing** at a rate of 2 cm/hour, how fast is the surface area changing when the cylinder is 150 cm tall?

Write your answer in a complete sentence, using units.

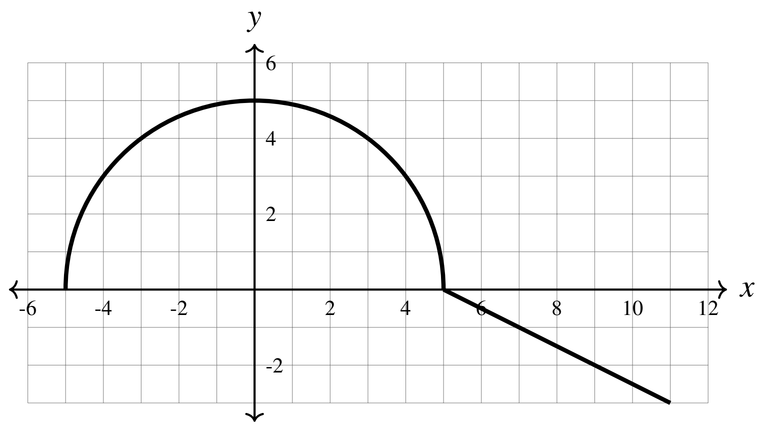
Recall that the surface area of a cylinder with radius r and height h is $S = 2\pi rh + 2\pi r^2$.

8. (12 points)

A portion of the function

$$f(x) = \begin{cases} \sqrt{25 - x^2} & -5 \le x \le 5 \\ -\frac{1}{2}(x-5) & x \ge 5 \end{cases}$$

is graphed below.



a. We want to approximate $\int_0^9 f(x)\,dx$ using **three** right-hand rectangles.

i. **Draw** the three right-hand rectangles on the graph. Lightly shade them in. Make sure I can tell where your rectangles are and try to be reasonably precise.

ii. Now **approximate** $\int_0^9 f(x)\,dx$ using the three right-hand rectangles. (Your answer should be a number.)

b. Use geometry to compute $\int_0^9 f(x)\,dx$ exactly. Show your work.

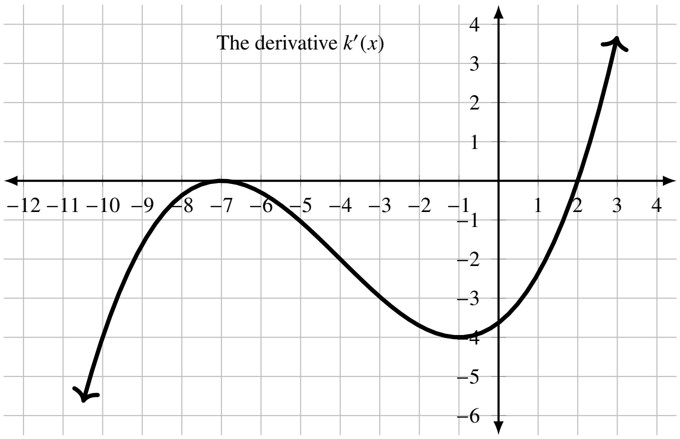
9. (10 points)

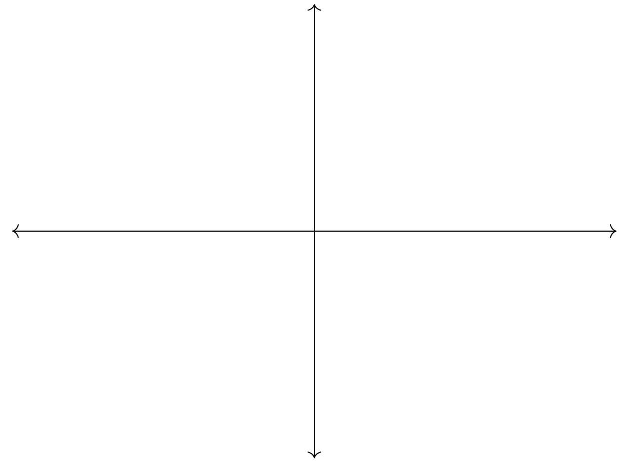
a. Determine $H(\theta) = \displaystyle\int \theta^{2/3} + (\sec(\theta))^2 + 5 \, d\theta$. (Give the most generic answer.)

b. Determine $G(x) = \displaystyle\int \frac{14x^3 + 2x + 1}{x^3} \, dx$. (Give the most generic answer.)

c. Find two different antiderivatives for the function $f(t) = 4e^t + \dfrac{1}{t}$.

$\boxed{\text{Extra Credit}}$ (5 points) The left-hand graph shows the $\boxed{\text{DERIVATIVE}}$ k' of some function k.





The following questions are about the original function $k(x)$, not the graphed $k'(x)$.

a. At what value(s) of x does k have a local maximum or minimum? If none, say so. **Write some words** to explain how you know that these x-values correspond to a local max or min (or not).

Local Max: $x =$ ________________ Local Min: $x =$ ________________

b. At what values of x does k have inflection points? If none, say so. **Write some words** to explain how you know that these x-values correspond to inflection points (or not).

Inflection points: $x =$ ________________________________

c. Sketch the graph of the original function $k(x)$ on the axes to the left of the derivative, using the fact that $k(-7) = 1$. Your sketch does not have to be to scale, but it should show the right information regarding increasing/decreasing/max/min/concavity/inflection points. Mark important values on the $x-$ and y-axes.