Fall 2024 **Math F251X**

Calculus I: Final Exam

Name: Solutions

Section:
- ☐ 9:15 (James Gossell)
- ☐ 11:45 (Jill Faudree)
- ☐ 11:45 (Leah Berman)
- ☐ Online (James Gossell)

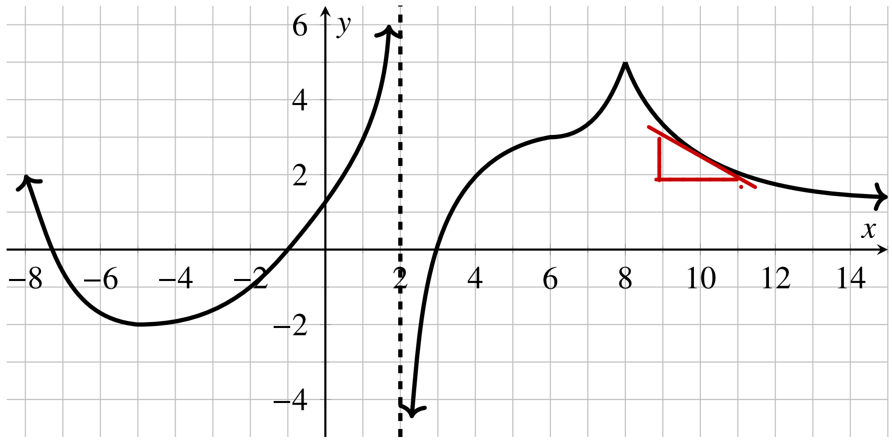
Rules:

- Partial credit will be awarded, but you must **show your work**.
- You may have a single handwritten $3'' \times 5''$ notecard, both sides.
- Calculators are **not allowed**.
- Place a box around your $\boxed{\text{FINAL ANSWER}}$ to each question where appropriate.
- Turn off anything that might go beep during the exam.
- You have two hours to complete the exam.

Problem	Possible	Score
1	16	
2	6	
3	10	
4	6	
5	8	
6	9	
7	11	
8	9	
9	16	
10	9	
Extra Credit	(5)	
Total	100	

1. (16 points)

Consider the graph of the function $f(x)$ shown below, and answer the following questions. You must give the most complete answer; if the value is infinite, write $+\infty$ or $-\infty$.



a. $\lim_{x\to 2^+} f(x) =$ $-\infty$

b. $\lim_{x\to 8} f(x) =$ 5

c. $\lim_{x\to 2^+} f'(x) =$ $+\infty$

d. List all x-values where the derivative $f'(x)$ is not defined. $x=2$, $x=8$

e. Estimate $f'(10) =$ $-\frac{1}{2}$. **Explain** how you computed your estimate.

I used the tangent line and triangle

f. List all intervals where $f'(x) > 0$. $(-5,2) \cup (2,6) \cup (6,8$

g. As what x-values does $f(x)$ have...(If none write "none").

A local maximum? $x=8$ A local minimum? $x=-5$.

h. The line $y = 1$ is a horizontal asymptote of $f(x)$. Fill in a statement about a limit that corresponds to this fact.

$$\lim_{\boxed{x\to\infty}} \boxed{f(x)} = \boxed{1}$$

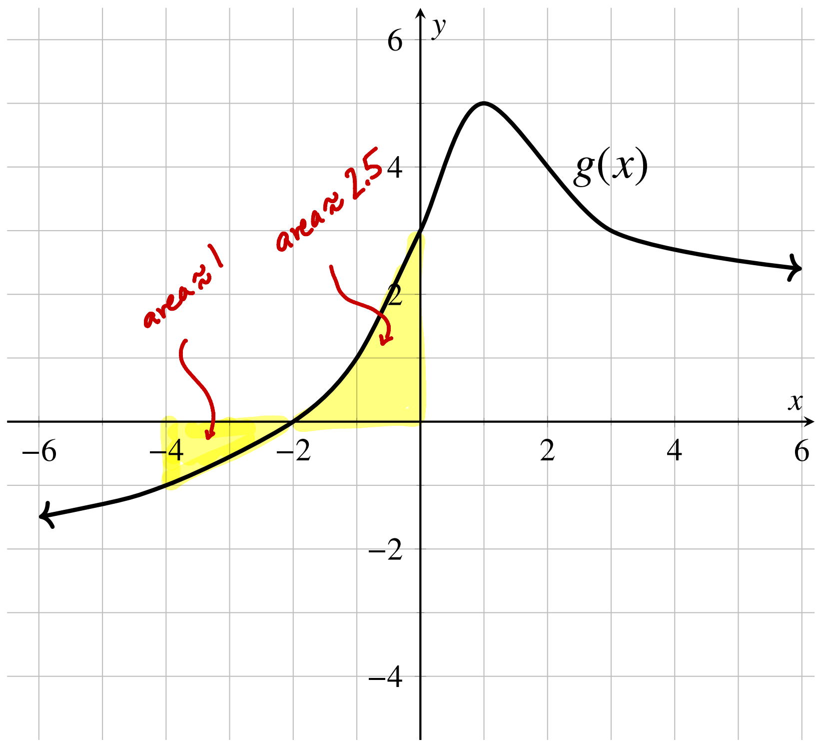
i. Does f have an(y) inflection point(s)? If so, list it/them, if not write "none."

Inflection point(s): $x =$ 6

j. On what interval(s) is $f(x)$ concave down? $(2,6)$

2. (6 points)

The graph of a function $g(x)$ is shown below.



Define a new function $A(x) = \int_{-4}^{x} g(t)\,dt$.

a. On the interval $(-4, 4)$, does $A(x)$ have a local maximum or a local minimum? If so, give the corresponding x-value and **explain** your answer. If not, explain why not.

A(x) has a local minimum at x = −2 because $A'(x) = g(x)$ switches from negative to positive.

b. Estimate $A(0) =$ 1.5 and clearly explain what calculation you did to arrive at that estimation. (You may want to draw on the graph as part of your explanation.)

$$A(0) = \int_{-4}^{-2} g(t)\,dt + \int_{-2}^{0} g(t)\,dt = -1 + 2.5 = 1.5$$

c. Determine $A'(2) =$ 4.

3. (10 points)

Compute the following **integrals**. Give the most general answer, and show your work. Clearly indicate any substitutions you use in such a way that someone else can follow your work.

a. $\int \left(3t^{\frac{2}{7}} - 8t^{-1} + e^{5t-4} + \sin\left(\frac{\pi}{6}\right)\right) dt$

$= 3 \cdot \frac{7}{9} t^{\frac{9}{7}} - 8\ln|t| + \frac{1}{5} e^{5t-4} + \frac{1}{2} t + C$

b. $\int x\cos(x^2) + \frac{1}{\sqrt{1-x^2}}\, dx$ $= \frac{1}{2}\sin(x^2) + \arcsin(x) + C$

4. (6 points)

Use linearization to estimate $\sqrt{98}$. Show your work, and write your answer as a single decimal or fraction.

Let $f(x) = \sqrt{x}$. Choose $a = 100$. $f'(x) = \frac{1}{2}x^{-1/2}$.

So $f(a) = f(100) = \sqrt{100} = 10$ and $f'(a) = f'(100) = \frac{1}{2} \cdot \frac{1}{\sqrt{100}} = \frac{1}{20}$

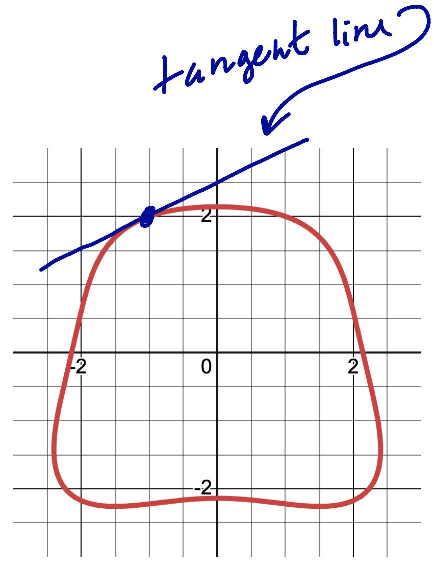
Now $L(x) = f(a) + f'(a)(x-a) = 10 + \frac{1}{20}(x - 100)$.

Finally, $\sqrt{98} = f(98) \approx L(98) = 10 + \frac{1}{20}(98-100) = 10 + \frac{1}{20}(-2)$

$= 10 - \frac{1}{10} = 9.9 = \frac{99}{10}$

5. (8 points)

Below is the graph of the curve $x^4 + y^4 + 2x^2y = 21$.



a. Find $\dfrac{dy}{dx}$ for the curve $x^4 + y^4 + 2x^2y = 21$.

$$4x^3 + 4y^3\frac{dy}{dx} + 4xy + 2x^2\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-(4xy + 4x^3)}{4y^3 + 2x^2}$$

$$= \frac{-(2xy + 2x^3)}{2y^3 + x^2}$$

b. **Write an equation** of the line tangent to the curve $x^4 + y^4 + 2x^2y = 21$ at the point $(-1, 2)$.

$$\left.\frac{dy}{dx}\right|_{(-1,2)} = \frac{-(2(-1)2 + 2(-1)^3)}{2(2)^3 + (-1)^2} = \frac{-(-4-2)}{16+1} = \frac{6}{17}$$

$$y - 2 = \frac{6}{17}(x+1) \quad \text{or} \quad y = 2 + \frac{6}{17}(x+1) \quad \text{or}$$

Tangent line equation: $y = \frac{6}{17}x + \frac{40}{17}$

c. **Draw** the tangent line on the figure above.

6. (9 points)

The Mars rover drops a rock off of a 61 foot cliff. While in free fall, the rock's **velocity** after t seconds is given by the function $v(t) = -12.2t$ feet per second.

a. Evaluate the integral $\int_0^2 v(t)\,dt$ and write a sentence to interpret its meaning in the context of the problem. Include units in your answer.

$$\int_0^2 -12.2t\,dt = -6.1t^2\Big]_0^2 = -6.1(4) = -24.4 \text{ feet.}$$

In the first two seconds, the rock falls 24.4 feet.

b. Write a function $h(t)$ that gives the rock's height above the ground after t seconds.

$h(0) = 61$

$h(t) = \int v(t)\,dt = \int 12.2t\,dt = 6.1t^2 + C$

So $61 = h(0) = 6.1(0)^2 + C$. So $C = 61$

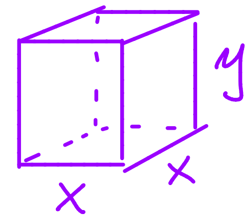
Answer: $h(t) = 6.1t^2 + 61$

c. What is the **acceleration** due to gravity on Mars? Include units in your answer.

$$a(t) = v'(t) = \frac{d}{dt}\left[12.2t\right] = 12.2 \text{ ft}/s^2$$

7. (11 points)

A box has a square base and an open top. The material for the base costs \$4 per square meter and the material for the sides costs \$1 per square meter. Suppose the width of the base of the box is x meters and its height is y meters.



a. What is the total cost, C, of the box?

$C = 4x^2 + 4xy$

b. What is the volume, V, of the box?

$V = x^2 y$

c. Suppose you have \$24 to spend on the materials for the box.

(i) Write the volume V as a function of one variable and pick a **domain** for this function.

$24 = 4x^2 + 4xy$

$24 - 4x^2 = 4xy$

$y = \frac{6}{x} - x$

$V(x) = x^2\left(\frac{6}{x} - x\right) = 6x - x^3$

domain: $(0, \infty)$ or $(0, \sqrt{24})$

(ii) Determine the **dimensions** of the box of **largest** possible volume that fits within your budget.

$V(x) = 6x - x^3$

$V'(x) = 6 - 3x^2 = 0$

$x^2 = 2$ or $x = \sqrt{2}$

(ignore negative root.)

Find y:

$y = \frac{6}{\sqrt{2}} - \sqrt{2} = 3\sqrt{2} - \sqrt{2} = 2\sqrt{2}$

dimensions (include units): $x =$ $\sqrt{2}$ m $y =$ $2\sqrt{2}$ m

(iii) **Justify** that your dimensions give the largest volume, using calculus. As part of your justification, **write the name** of the test you are applying (first derivative test, second derivative test, closed interval/extreme value theorem, some other test).

First Derivative Test (only 1 crit #.)

$+ + + +$ 0 $- - - - -$ ← sign of V'

0 1 $\sqrt{2}$ 10

$V'(1) > 0$

$V'(10) < 0$

So we have a max at $x = \sqrt{2}$

8. (9 points)

Compute the following **limits**. Show your work clearly. Make sure you use **limit notation** where required; an answer that does not use proper notation will not receive full credit. Use = to show things are equal. If you use L'Hôpital's rule, write $\stackrel{H}{=}$ or $\stackrel{L'H}{=}$ to indicate where you are applying it.

a. $\displaystyle\lim_{\theta\to\pi}\frac{1+\cos(\theta)}{\sin^2(\theta)} = \frac{1+(-1)}{(0)^2} = \frac{0}{0}$ Apply L'Hop

$\displaystyle\stackrel{H}{=} \lim_{\theta\to\pi}\frac{-\sin(\theta)}{2\sin(\theta)\cos(\theta)} = \lim_{\theta\to\pi}\frac{-1}{2\cos(\theta)} = \frac{-1}{2(-1)} = \frac{1}{2}$

b. $\displaystyle\lim_{h\to 0}\frac{\ln(e+h)-1}{h} = \frac{\ln(e)-1}{0} = \frac{0}{0}$ apply L'Hop

$\displaystyle\stackrel{H}{=} \lim_{h\to 0}\frac{\frac{1}{e+h}\cdot 1}{1} = \frac{\frac{1}{e}}{1} = \frac{1}{e}$

c. $\displaystyle\lim_{x\to\infty}\left(\frac{-10x^3+7x^2}{2x^3+3x-5}\right)\frac{\left(\frac{1}{x^3}\right)}{\left(\frac{1}{x^3}\right)} = \lim_{x\to\infty}\frac{-10+\frac{7}{x}}{2+\frac{3}{x^2}-\frac{5}{x^3}} = \frac{-10}{2} = -5$

9. (16 points)

At 3:00 pm ($t = 0$), workers discover a break in an oil pipeline. The rate at which oil is flowing from the break is given by $r(t) = \dfrac{10t}{4+t^2}$, where r is measured in gallons per hour and t is measured in hours.

a. Find $r(1)$. Write a sentence that someone who has not taken calculus could understand to explain the meaning of $r(1)$. Include units.

$$r(1) = \frac{10}{4+1} = 5 \text{ gal/hr}$$

At 4:00pm oil is leaking at a rate of 5 gallons per hour.

b. Determine how much oil flowed out of the pipeline between 3:00pm and 5:00pm. Include units in your answer.

$$\int_0^2 \frac{10t}{4+t^2} = 5\ln(4+t^2)\Big|_0^2 = 5(\ln(8)-\ln(4)) = 5\ln(2) \text{ gallons}$$

c. At 4:00pm, is the rate of flow increasing, decreasing or staying the same? Explain how you know.

4:00 pm means $t=1$.

$$r'(t) = \frac{(4+t^2)(10)-(10t)(2t)}{(4+t^2)^2} = \frac{40-10t^2}{(4+t^2)^2}; \quad r'(1) = \frac{30}{5^2} > 0$$

At 4:00pm, the rate of flow is increasing because $r'(1) > 0$.

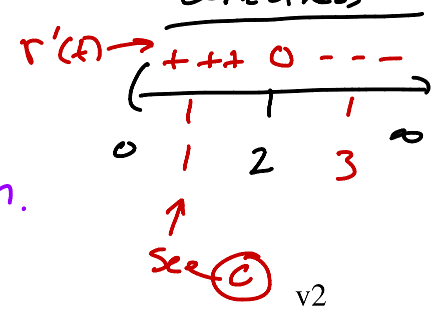
d. At what time is the rate of flow of the oil at its maximum? (Give your answer as a time.)

$r'(t) = 0$ when $40 - 10t^2 = 0$ or $t = \pm 2$.

(ignore the negative root)

$r(t)$ is maximized when $t = 2$, at 5:00pm.

quick check of correctness



v2

10. (9 points)

Compute the following **derivatives**. Show your work. You do NOT need to simplify your answer. Your answer should start $f'(x)$, $\frac{df}{dx}$ etc.

a. $h(x) = 4\sec\left(5x^{-2}\right)$

$$h'(x) = 4\sec(5x^{-2})\tan(5x^{-2})\left(5(-2)x^{-3}\right)$$

$$= -40x^{-3}\sec(5x^{-2})\tan(5x^{-2})$$

b. $f(x) = x^{-3} + \sqrt[3]{(2x)} - 3^x + 3$

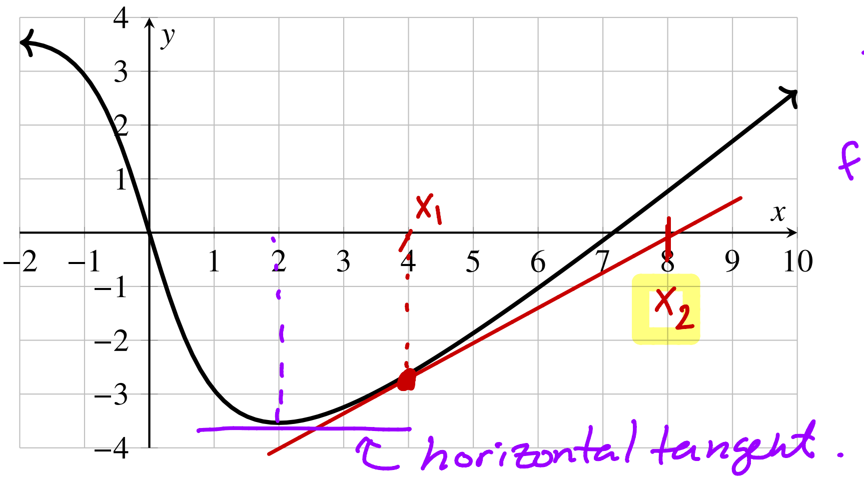
$$f'(x) = -3x^{-4} + \frac{1}{3}(2x)^{-2/3}(2) - (\ln 3)3^x$$

c. $g(x) = \ln\left(\frac{x^6 e^x}{\sqrt{x}}\right) = \ln\left(x^{\frac{11}{2}}e^x\right) = \frac{11}{2}\ln x + x$

$$g'(x) = \frac{11}{2x} + 1$$

alt: $g'(x) = \frac{\sqrt{x}}{x^6 e^x} \cdot \left(\frac{\sqrt{x}(6x^5e^x - \frac{1}{2}x^{-1/2}x^6e^x)}{(\sqrt{x})^2}\right)$

Extra Credit (5 points) A portion of the graph of the function $f(x) = x - 5\arctan(x)$ is shown below.



$f'(x) = 1 - \frac{5}{1+x^2}$

$f'(2) = 1 - \frac{5}{1+2^2} = 0$

a. Suppose Newton's method is used to find an approximate solution to $f(x) = 0$ from an initial guess of $x_1 = 4$. **Sketch** on the graph how the next approximation x_2 will be found, **labeling** its location on the x-axis.

b. If your starting guess is $x_1 = 4$, **compute** x_2. Show your work. **You do not need to simplify completely**, but your answer should be in a form where typing it into a calculator would compute a numerical value.

$f'(x) = 1 - \frac{5}{1+x^2}$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 4 - \left(\frac{x + 5\arctan(4)}{1 - \frac{5}{1+4^2}}\right)$

c. What happens if you try to apply Newton's method with a starting guess of $x_1 = 2$?

The method will fail because $f'(2) = 0$.