Fall 2024 **Math F251X**

Calculus 1: Midterm 1

Name: Solutions

Section: ☐ **9:15am (James Gossell)**
☐ **11:45am (Jill Faudree)**
☐ **11:45am (Leah Berman)**
☐ **async (James Gossell)**

Rules:

- Partial credit will be awarded, but you must show your work.
- You may have a single handwritten $3'' \times 5''$ notecard, both sides.
- Calculators are **not** allowed.
- Place a box around your $\boxed{\text{FINAL ANSWER}}$ to each question where appropriate.
- Turn off anything that might go beep during the exam.

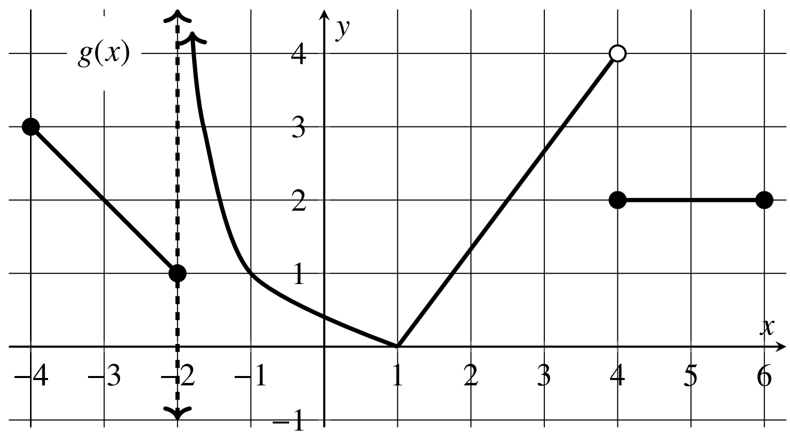
Good luck!

Problem	Possible	Score
1	14	
2	15	
3	10	
4	15	
5	8	
6	12	
7	6	
8	8	
9	12	
Extra Credit	(5)	
Total	100	

v2

1. (14 points)

The graph of a function $g(x)$ is shown below. The domain of the function is $[-4, 6]$. Use the graph to answer each question below. If the limit is infinite, indicate that with ∞ or $-\infty$. If the value does not exist or is undefined, write DNE.



a. $\lim_{x\to -2^-} g(x) =$ 1

b. $\lim_{x\to -2^+} g(x) =$ $+\infty$

c. $g(1) =$ 0

d. $\lim_{x\to 1} g(x) =$ 0

e. $\lim_{x\to 1^-} g'(x) =$ $-1/2$

Note this is asking about the limit of the derivative.

f. $g'(1) =$ DNE

g. $g'(2) =$ $4/3$

h. $\lim_{x\to 4} g(x) =$ DNE

i. $g'(5) =$ 0

j. List all **x-values** in the set $(-4, 6)$ where the function $g(x)$ is **not** continuous.

$x =$ $-2, 4$

k. Fill in the empty boxes to make a true sentence.

The function $g(x)$ has a vertical asymptote with equation $\boxed{x = -2}$ because

$$\lim_{\boxed{x\to -2^+}} g(x) = \boxed{+\infty}.$$

2. (15 points)

Compute the following limits. Show your work. Use limit notation where necessary; you will be graded both on your computation and on your correct use of notation.

Give the most complete answer. Specifically, if the limit approaches $+\infty$ or $-\infty$, write that. The answer DNE is insufficient in this case.

a. $\displaystyle\lim_{x\to 3}\frac{15-5x}{x^2-10x+21} = \lim_{x\to 3}\frac{5(3-x)}{(x-3)(x-7)} = \lim_{x\to 3}\frac{-5\cancel{(x-3)}}{\cancel{(x-3)}(x-7)}$

$\displaystyle\frac{15-15}{9-30+21} = \frac{0}{0}$

$\displaystyle = \lim_{x\to 3}\frac{-5}{x-7} = \frac{-5}{-3-7} = \frac{5}{10} = \boxed{\frac{1}{2}}$

So factor and cancel!

b. $\displaystyle\lim_{x\to 0}\frac{2\cos x}{\sin x-1} = \frac{2\cos(0)}{\sin(0)-1} = \frac{2}{0-1} = \boxed{-2}$

c. $\displaystyle\lim_{x\to 4^+}\frac{\overset{+}{2x+17}}{\underset{-}{(x-4)(x-5)}} = \boxed{-\infty}$

$\displaystyle\frac{8+17}{0(-1)}$

as $x\to 4^+$, $2x+17\to 25$, $\underset{+}{(x-4)}\underset{-}{(x-5)}\to 0^-$

So approaches $\pm\infty$

d. $\displaystyle\lim_{x\to 4}\frac{\frac{x}{4}-\frac{4}{x}}{x-4} = \lim_{x\to 4}\left(\frac{1}{x-4}\right)\left(\frac{x^2-16}{4x}\right) = \lim_{x\to 4}\frac{\cancel{(x-4)}(x+4)}{\cancel{(x-4)}(4x)}$

common denominator

$\displaystyle = \lim_{x\to 4}\frac{x+4}{4x} = \frac{4+4}{4\cdot 4} = \frac{8}{16} = \boxed{\frac{1}{2}}$

3. (10 points)

Consider the function

$$f(x) = \sqrt{2-x}.$$

Use the **limit definition of the derivative** to compute the derivative, and show your work using all appropriate notation. No credit will be awarded for using other methods. You must write limits where necessary to receive full credit.

The limit definition of the derivative:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

$$f'(x) = \lim_{h\to 0} \left(\frac{\sqrt{2-(x+h)} - \sqrt{2-x}}{h} \right) \cdot \left(\frac{\sqrt{2-(x+h)} + \sqrt{2-x}}{\sqrt{2-(x+h)} + \sqrt{2-x}} \right)$$

$$= \lim_{h\to 0} \frac{\cancel{2}\cancel{-x}-h \cancel{-2}\cancel{+x}}{h\left(\sqrt{2-(x+h)} + \sqrt{2-x}\right)} = \lim_{h\to 0} \frac{-\cancel{h}}{\cancel{h}\left(\sqrt{2-x-h} + \sqrt{2-x}\right)}$$

$$= \lim_{h\to 0} \frac{-1}{\sqrt{2-x-h} + \sqrt{2-x}} = \boxed{\frac{-1}{2\sqrt{2-x}}}$$

4. (15 points)

For each of the following functions, compute the derivative. **You do not need to simplify your answers.** Your answer must begin with $f'(x)$ or similar notation, as appropriate to the problem.

a. $f(x) = x^{3.2} + \dfrac{\sin(x)}{5} + \dfrac{\sqrt[3]{x}}{3} + \sqrt{3}$ $= x^{3.2} + \frac{1}{5}\sin(x) + \frac{1}{3}x^{1/3} + \sqrt{3}$

$$\boxed{f'(x) = 3.2x^{2.2} + \frac{1}{5}\cos(x) + \frac{1}{9}x^{-2/3}}$$

b. $g(y) = y\cos(y)$ (product rule)

$$\boxed{g'(y) = 1\cdot\cos(y) + y(-\sin(y)) = \cos(y) - y\sin(y)}$$

c. $h(\theta) = \dfrac{\theta^4}{\theta^3 + 6}$ (quotient rule)

$$h'(\theta) = \frac{4\theta^3(\theta^3+6) - \theta^4(3\theta^2)}{(\theta^3+6)^2} = \frac{4\theta^6 + 24\theta^3 - 3\theta^6}{(\theta^3+6)^2}$$

$$= \frac{\theta^6 + 24\theta^3}{(\theta^3+6)^2} = \boxed{\frac{\theta^3(\theta^3+24)}{(\theta^3+6)^2}}$$

5. (8 points)

Hopewell Cape, off the east coast of Canada, is known to have one of the highest tides in the world. The function $D(t)$ models the water depth at Hopewell Cape over a day, and is given by

$$D(t) = 7 + 5\cos(\,0.503(t - 6.75)\,)$$

In this function, D is water depth in meters and t is measured in hours after midnight.

a. Explain the meaning of the fact that $\boxed{D(4) = 7.9}$ in the context of the problem. Make sure your explanation is one that a typical precalculus student could understand, and don't forget to include units.

At 4:00 am, the depth of the water is 7.9 meters.

b. Explain the meaning of the fact that $\boxed{D'(4) = 2.5}$ in the context of the problem. Make sure your explanation is one that a typical precalculus student could understand. (Do not use the word **derivative**!) And don't forget to include units.

At 4:00 am, the depth of the water is increasing at a rate of 2.5 meters per hour.

c. Using the facts that $\boxed{D(4) = 7.9 \text{ and } D'(4) = 2.5}$, estimate the water depth at 5:00 AM.

$\underset{\text{5:00am}}{\text{depth at}} = D(5) \approx 7.9 + 2.5 = 10.4$ meters.

d. Do you expect $D'(t)$ to ever take on negative values? Explain your answer.

Yes. We do expect $D'(t) < 0$ for some t-values.
We expect the tide to go out. So water depth will drop.

Also, we know $D(t)$ is a transformation of the cosine function which has negative slope.

6. (12 points)

Consider the function $g(x) = \frac{1}{x} + 8x^2$. $= x^{-1} + 8x^2$

a. Find $g'(x)$. (You do not need to use the limit definition of the derivative to answer this problem.)

$g'(x) = -x^{-2} + 16x = \frac{-1}{x^2} + 16x$

b. Write an equation for the tangent line to $g(x)$ when $x = 1$.

$g'(1) = -1+16 = 15$

$g(1) = 1+8 = 9$

$y - 9 = 15(x-1)$ or

$y = 9 + 15(x-1)$ or

$y = 15x - 6$

c. Determine the x-coordinates of all points on the graph of $g(x)$ where there is a **horizontal** tangent line. Show your work.

Set $g'(x) = 0$.

So $-\frac{1}{x^2} + 16x = 0$

$16x = \frac{1}{x^2}$ or $x^3 = \frac{1}{16}$

So $x = \sqrt[3]{1/16} = \frac{1}{2\sqrt[3]{2}}$

7. (6 points)

Consider the piecewise-defined function given below. The value c is a constant whose value you do not yet know.

$$f(x) = \begin{cases} cx^2 + 3 & \text{if } x \geq 3 \\ \frac{2}{x-2} & \text{if } x < 3 \end{cases}$$

a. What should c be so that $f(x)$ is continuous at $x = 3$? Justify your answer with limits, and show your work.

$\lim_{x \to 3^-} \frac{2}{x-2} = \frac{2}{3-2} = 2.$

So we need $c(3^2) + 3 = 2$

$9c = -1$

$c = -1/9$

b. Identify any other points where $f(x)$ is not continuous. Explain how you know there is a discontinuity. Identify the type of discontinuity.

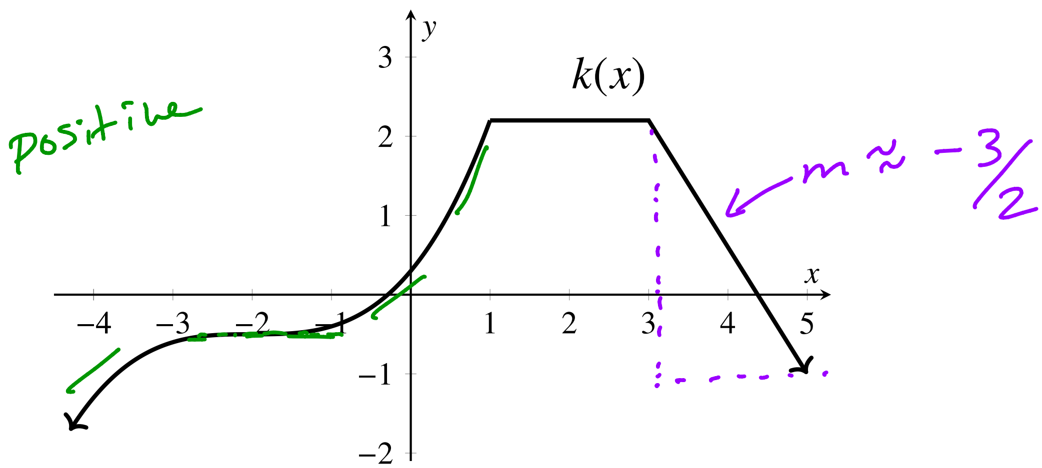
f is not continuous at $x=2$.

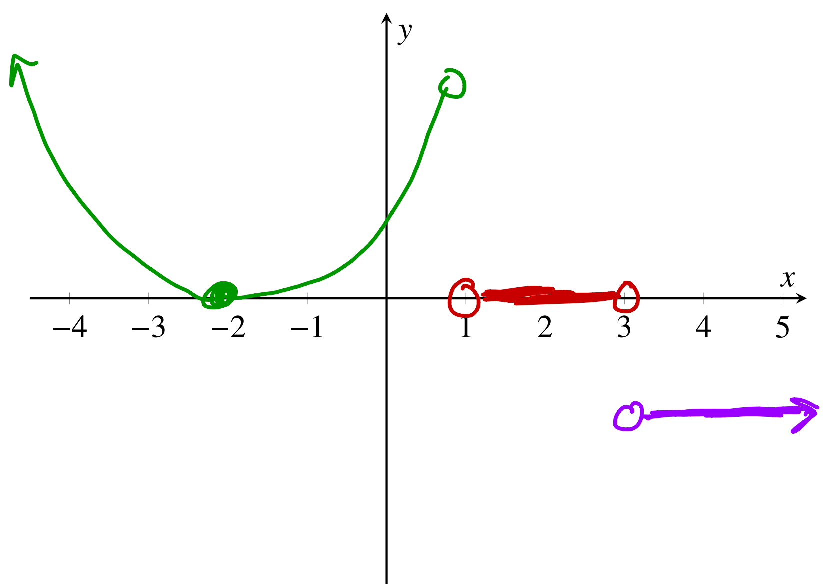
$\lim_{x \to 2^+} \frac{2}{x-2} = +\infty.$ It is an infinite discontinuity.

8. (8 points)

The graph of a function $k(x)$ is shown below. On the set of axes below the graph, sketch the derivative, $k'(x)$.

Indicate any asymptotes the derivative might have using dashed lines, and indicate any points where the derivative is undefined using open circles.





9. (12 points)

The position function $s(t) = 8t - t^3$ gives the position in miles of a freight train running along a straight east-west railroad track where east is the positive direction and t is measured in hours. Assume $t \geq 0$.

a. Find expressions for velocity and acceleration of the train at time t.

$v(t) = s'(t) = 8 - 3t^2$

$a(t) = s''(t) = -12t$

b. What is the **average** velocity of the train in the first two hours from $t = 0$ to $t = 2$? Include units.

$t=0: \ s(0) = 0$

$t=2: \ s(2) = 8\cdot 2 - 2^3$
$= 8$

avg vel $= \frac{8-0}{2-0} = 4 \text{ mi/hr}$

c. Determine the direction the train is traveling at the start of the trip when $t = 0$.

at $t=0$ $v(0) = 8 > 0$.

So the train is traveling east.

d. At $t = 1$ is the train speeding up or slowing down? Justify your answer.

$t=1, \ s'(1) = 8 - 3 = 5 > 0$

$s''(1) = -12 < 0$

The train is slowing down

e. Determine the time (or times) at which the train changes direction or explain why this does not happen.

$s'(t) = 0$ when $8 - 3t^2 = 0$

$3t^2 = 8$

$t = \sqrt{\frac{8}{3}} = \frac{2\sqrt{2}}{\sqrt{3}}$ hours

10. (Extra Credit: 5 points)

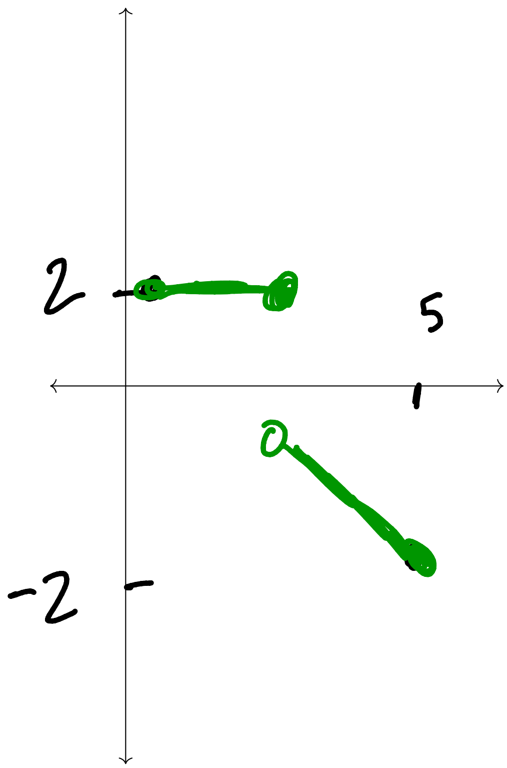
a. On the given set of axes, draw a function $f(x)$ that satisfies all of the following properties. Label important points on the x- and y-axes.

(i) $f(0) = 2$

(ii) $f(5) = -2$

(iii) The domain of f is the entire interval $[0, 5]$.

(iv) The graph of $f(x)$ never crosses the x-axis.



b. The Intermediate Value Theorem says

> Let f be continuous over a closed, bounded interval $[a, b]$. If z is any real number between $f(a)$ and $f(b)$, then there is a number c in $[a, b]$ satisfying $f(c) = z$.

Does your function contradict the statement of the Intermediate Value Theorem (IVT)? Does it agree with the statement of the IVT?

If it contradicts the statement of the IVT, explain how. If it agrees with the statement of the IVT, explain how.

The graph does not contradict the IVT because the graph is not continuous, something IVT requires.