Fall 2024 **Math F251X**

Calculus 1: Midterm 2

Name: Solutions

Section:
☐ **9:15am (James Gossell)**
☐ **11:45am (Jill Faudree)**
☐ **11:45am (Leah Berman)**
☐ **async (James Gossell)**

Rules:

- Partial credit may be awarded, but you must show your work.
- You may have a single handwritten 3″ × 5″ notecard, both sides.
- Calculators are **not** allowed.
- Place a box around your $\boxed{\text{FINAL ANSWER}}$ to each question where appropriate.
- Turn off anything that might go beep during the exam.

Good luck!

Problem	Possible	Score
1	12	
2	12	
3	10	
4	11	
5	12	
6	12	
7	12	
8	9	
9	10	
Extra Credit	5	
Total	100	

1. (12 points)

Evaluate the following limits. **Show your work**, uncluding appropriate use of limit notation. If you use L'Hôpital's rule, you must indicate where you are using it by writing $\overset{H}{=}$ or $\overset{L'H}{=}$ or something similar. Use ∞ or $-\infty$ where appropriate, and if the limit does not exist, write DNE and provide a justification.

a. $\lim\limits_{t\to 2} \dfrac{e^{t-2}-t+1}{t^2-4t+4} = \dfrac{e^{0}-2+1}{4-8+4} = \dfrac{0}{0}$

$$\overset{\textcircled{h}}{=} \lim_{t\to 2} \frac{e^{t-2}-1}{2t-4} \overset{\textcircled{h}}{=} \lim_{t\to 0} \frac{e^{t-2}}{2} = \frac{1}{2}$$

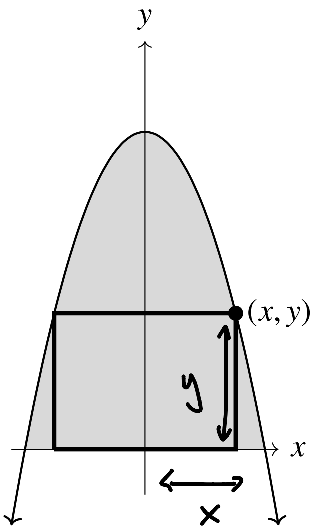
↑ form $\frac{0}{0}$

b. $\lim\limits_{x\to\infty} \left(\dfrac{3x-x^4}{3x^4-4x^3-1}\right) \dfrac{\cdot\, ^1/_{x^4}}{\cdot\, ^1/_{x^4}} = \lim\limits_{x\to\infty} \dfrac{\frac{3}{x^3}-1}{3-\frac{4}{x}-\frac{1}{x^4}} = \dfrac{-1}{3}$

c. $\lim\limits_{\theta\to 0} \dfrac{4\sin(\theta)-4}{1-\theta-e^{\cos(\theta)}} = \dfrac{4\cdot 0-4}{1-0-e^{1}} = \dfrac{-4}{1-e}$

2. (12 points)

We want to determine the dimensions of the rectangle of maximum area that is inscribed between the parabola $y = 7 - x^2$ and the x-axis. Assume the base of the rectangle is on the x-axis. (See figure below; the rectangle should be inside the shaded area.)



a. Find an expression for the area A of the rectangle as a function of one variable.

$$A = 2xy = 2x(7-x^2)$$
$$= 14x - 2x^3$$

b. State the appropriate domain of the area function given the context of the problem.

$[0, \sqrt{7}]$

c. Use Calculus to determine where the area is maximized. Justify your conclusion with work.

Find crit. #'s: $A'(x) = 14 - 6x^2 = 0$. So $x^2 = \frac{14}{6} = \frac{7}{3}$.

So $x = {}^{+}\sqrt{7/3}$ ($x = -\sqrt{7/3}$ is not in domain)

Check crit# is a max: (There are 3 ways.)

① Closed-interval method

x	A(x)
0	0
$\sqrt{7}$	0
$\sqrt{7/3}$	$2\sqrt{7/3}\left(7-\frac{7}{3}\right)$ $= \frac{28\sqrt{7}}{3\sqrt{3}}$

② 2^{nd} Der. test

$A''(x) = -12x$

$A''\left(\sqrt{7/3}\right) = -12\left(\sqrt{7/3}\right) < 0$

So A is cc down ✓

So max @

$x = \sqrt{7/3}$

③ 1^{st} der. test

$$\begin{array}{ccc} +++ & 0 & ---- \\ 0 & \sqrt{7/3} & \sqrt{7} \end{array}$$

$A'(0) = 14 > 0$

$A'(\sqrt{7}) = 14 - 42 < 0$

So max at

$x = \sqrt{7/3}$

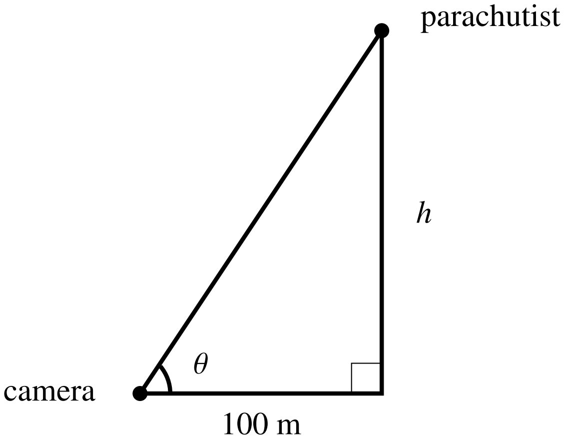
d. Answer the question:
The dimensions of the rectangle with largest area are width $= 2\sqrt{7/3}$, height $= \frac{14}{3}$

$y = 7 - x^2 = 7 - \frac{7}{3} = \frac{14}{3}$

3. (10 points)

A camera at ground level is 100 meters from the landing site of a parachutist who is landing vertically. Let h be the height of the parachutist above the ground and let θ be the angle of elevation formed between the camera lens and the ground. (See figure.)

a. Find an equation relating h and θ.



$$\tan\theta = \frac{h}{100} \qquad \text{or}$$

$$\theta = \arctan\left(\frac{1}{100}h\right)$$

b. Suppose the height of the parachutist decreases at a constant rate of 5 meters per second. At what rate does the angle θ decrease when the parachutist is 200 meters in the air? **Answer the question with a complete sentence, including units.**

$\frac{dh}{dt} = -5$, Find $\frac{d\theta}{dt}$ when $h = 200$

$$\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{1}{100}h\right)^2}\cdot\frac{1}{100}\cdot\frac{dh}{dt} = \frac{1}{1+2^2}\cdot\frac{1}{100}\cdot(-5) = -\frac{1}{100}$$

The angle is decreasing at a rate of $\frac{1}{100}$ rad/s.

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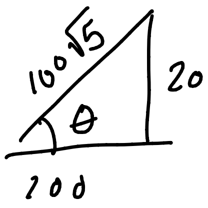
alternate approach $\sec^2\theta\,\frac{d\theta}{dt} = \frac{1}{100}\frac{dh}{dt}$, $5\cdot\frac{d\theta}{dt} = \frac{1}{100}(-5)$.

when $h = 200$ m



$$\sec\theta = \frac{100\sqrt{5}}{100} = \sqrt{5}$$

So $\frac{d\theta}{dt} = \frac{1}{100}$ rad/s

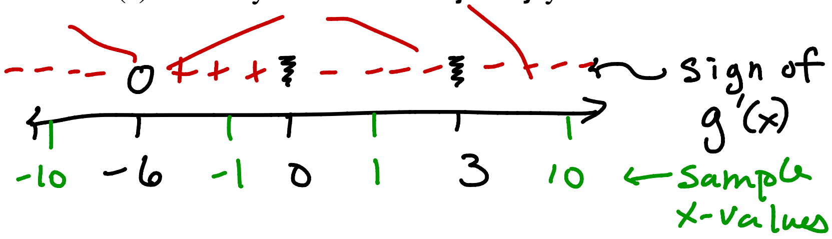
4. (11 points)

Consider the function $g(x) = \frac{x^{2/3}}{x-3}$. After simplification, $g'(x) = \frac{-(x+6)}{3x^{1/3}(x-3)^2}$.

a. What are the critical numbers of $g(x)$?

$x = -6, 0, \cancel{3}$ ($x=3$ is not in the domain.)

b. At what x-values does $g(x)$ have local maximum(s)? At what x-values does g(x) have local minimum(s)? Clearly show work to **justify** your answers.



$g'(-10) = \frac{-(-)}{-(+)} = -$

$g'(1) = \frac{-(+)}{+} = -$, $g'(10) = \frac{-}{+} = -$

$g'(-1) = \frac{-(+)}{-(+)} = +$

Local maximum(s): $x =$ 0 Local minimum(s): $x =$ -6

(If none, write "none".)

c. Does $g(x)$ have any horizonal asympotes? If it does, write the equations of any horizontal asymptote(s) of $g(x)$, and justify each answer by writing a limit. If it doesn't, explain why $g(x)$ does not have any horizontal asymptotes and write "none".

$$\lim_{x\to+\infty} \frac{x^{2/3}}{x-3} \frac{\cdot 1/x}{\cdot 1/x} = \lim_{x\to+\infty} \frac{\frac{1}{x^{1/3}}}{1-\frac{3}{x}} = 0$$

$$\lim_{x\to-\infty} \frac{x^{2/3}}{x-3} = 0$$

Horizontal asymptote equation(s): $y=0$

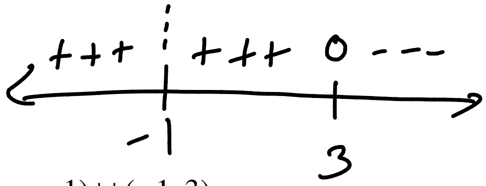
(If none, write "none".)

5. (12)

Sketch a graph of a function $f(x)$ that satisfies all of the following properties.

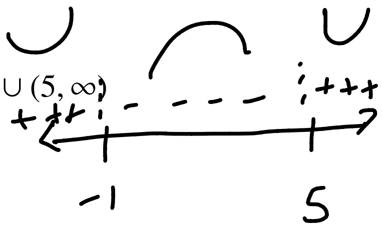
After drawing the graph:

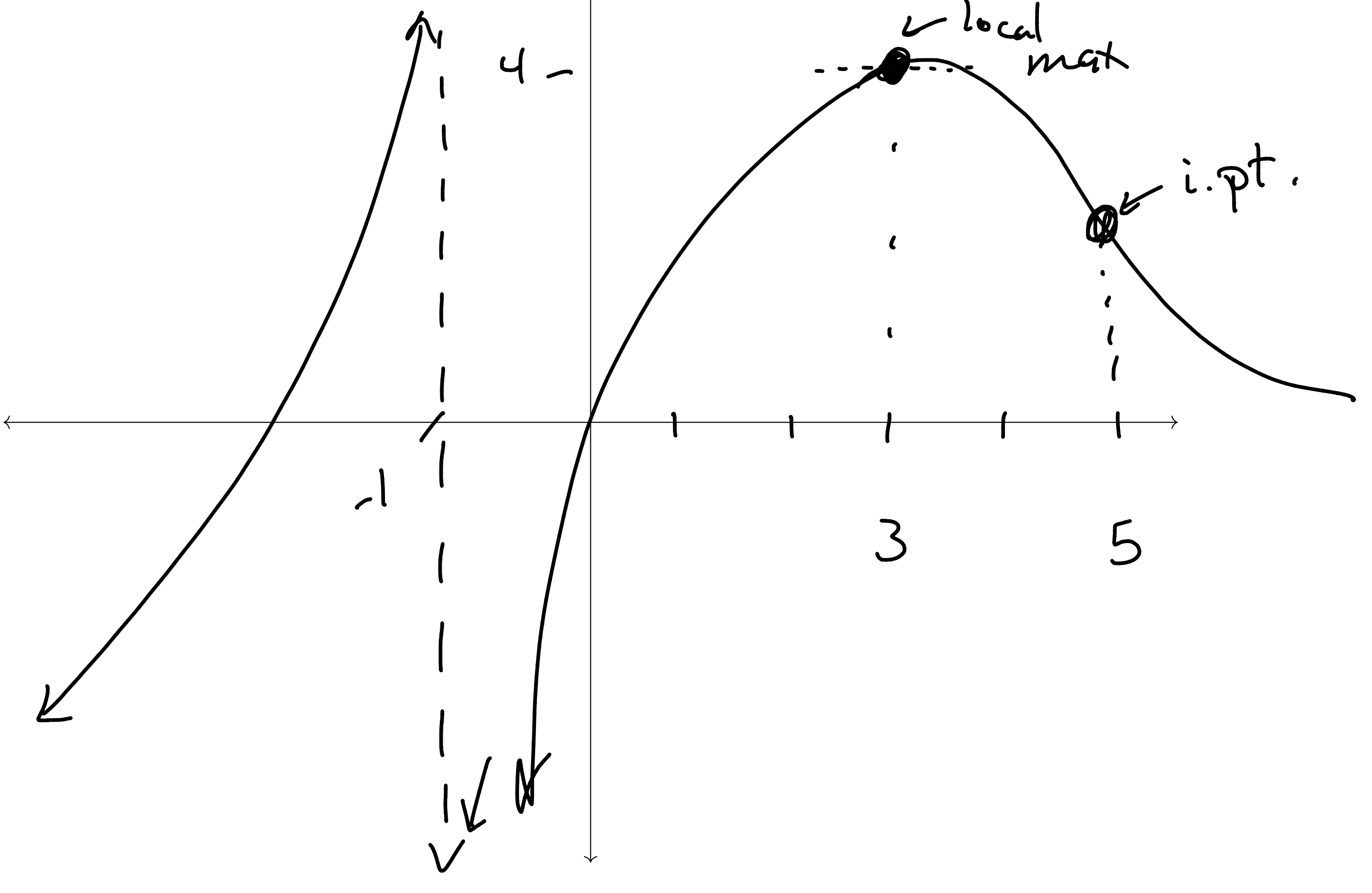
- **Label** on the graph the following things, if they exist, by drawing a point on the graph and labeling: any local maximums by writing LOCAL MAX, local minimums by writing LOCAL MIN, inflection points by writing IP
- Draw any horizontal and vertical asymptotes with dashed lines and **label** them with their equation.
- Mark any important x-values and y-values on the x- and y-axes.



Properties:

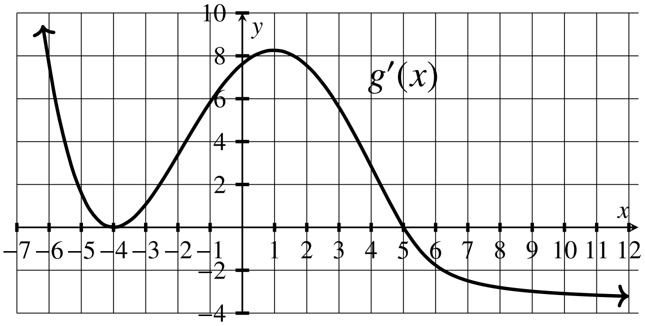
- Domain is $(-\infty, -1) \cup (-1, \infty)$
- $f(3) = 4$ and $f'(3) = 0$
- $\lim_{x \to -1^+} f(x) = -\infty$
- $f'(x) > 0$ on $(-\infty, -1) \cup (-1, 3)$
- $f'(x) < 0$ on $(3, \infty)$
- $f''(x) > 0$ on $(-\infty, -1) \cup (5, \infty)$
- $f''(x) < 0$ on $(-1, 5)$





6. (12 points)

The graph shown below is the graph of the $\boxed{\textbf{derivative } g'(x)}$ of a function $g(x)$. Answer the following questions about the **original** function $g(x)$.



a. Determine the critical numbers of $g(x)$. (Notice that $g(x)$ is $\boxed{\textbf{not}}$ shown on the graph!)

$x = -4, 5$

b. Determine the intervals where g is **increasing** and where g is **decreasing**. If none write "none".

Increasing: $(-\infty, 5)$ Decreasing: $(5, \infty)$

c. Fill in the blanks (if none, write "none"):

$g(x)$ has (a) local maximum(s) at $x =$ 5 and (a) local minimum(s) at $x =$ none.

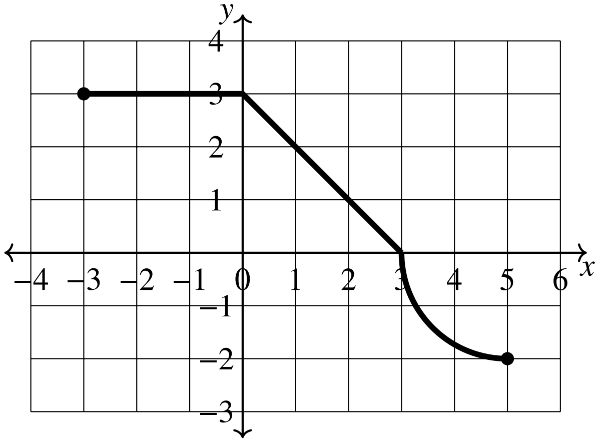
d. Find all intervals where g is **concave up** and where g is **concave down**. (If none write "none".)

Concave up: $(-4, 1)$ Concave down: $(-\infty, -4) \cup (1, \infty)$

e. Fill in the blanks: $g(x)$ has (an) inflection point(s) at $x =$ $-4, 1$. (If none, write "none".)

7. (12 points)

The graph of the function $G(x)$ is below. The function $G(x)$ has domain $[-3, 5]$ and the portion of the graph on the interval $[3, 5]$ is one quarter of a circle of radius 2 centered at the point $(5, 0)$.



a. Determine $\int_{-1}^{3} G(x)\,dx$. $= 7.5$

b. Determine $\int_{-3}^{5} G(x)\,dx$. $= 9 + 4.5 - \frac{1}{4} \cdot \pi \cdot 2^2 = 13.5 - \pi$

c. Determine $\int_{-3}^{5} 2G(x) + 5\,dx$ (Hint: Use part (b) above.)

$= 2(13.5 - \pi) + 5 \cdot 8$

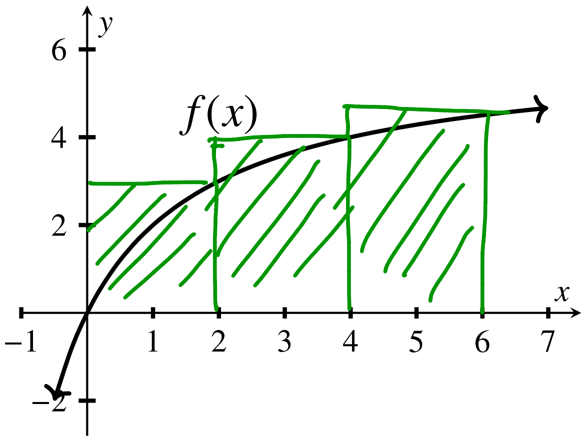
$= 27 - 2\pi + 40 = 67 - 2\pi$

8. (9 points)

Consider the function

$$f(x) = \frac{6x}{x+2}.$$

A portion of the graph of this function is shown to the right.



a. Compute R_3 on the interval $[0,6]$. That is, approximate $\int_0^6 f(x)\,dx$ using 3 right-hand rectangles. **Draw the rectangles on the graph.**

$R_3 = 2\left(f(2)+f(4)+f(6)\right) = 2\left(\frac{12}{4}+\frac{24}{6}+\frac{36}{8}\right) = 2(3+4+4.5) = 23$

b. List **two distinct strategies** to compute a more accurate approximation of $\int_0^6 f(x)\,dx$.

- more rectangles
- use mid points

9. (10 points)

Evaluate the indefinite integrals below. (Give the most generic answer.)

a. $\int \left(4x^4 + \sin(x) - e^x + \sqrt{2}\right)\,dx = \frac{4}{5}x^5 - \cos(x) - e^x + \sqrt{2}\,x + C$

b. $\int \frac{1 + x^{\frac{2}{3}} + x^3}{x}\,dx = \int \left(x^{-1} + x^{-\frac{1}{3}} + x^2\right) dx$

$= \ln|x| + \frac{3}{2}x^{2/3} + \frac{1}{3}x^3 + C$

Extra Credit (5 points)

A population of bacteria can be modeled by the function $P(t) = t^{k/t}$, where t is time, measured in hours, P is the number of bacteria, measured in thousands, and k is a fixed positive constant.

a. Compute $\lim_{t\to\infty} P(t)$.

form ∞^0

$$\lim_{t\to\infty} t^{k/t} = \boxed{e^0} = 1$$

take natural log of P(t).

$$\lim_{t\to\infty} \frac{k\ln t}{t} \overset{\textcircled{h}}{=} \lim_{t\to\infty} \frac{k\cdot\frac{1}{t}}{1} = \frac{0}{1} = 0$$

form $\frac{\infty}{\infty}$

b. Interpret this limit by writing a complete sentence, including units, using the context of the model.

In the long run, the population of bacteria stabilizes at 1 thousand bacteria.