Fall 2024 **Math F251X**

Calculus 1: Midterm 2

Name: Solutions

Section:
☐ **9:15am (James Gossell)**
☐ **11:45am (Jill Faudree)**
☐ **11:45am (Leah Berman)**
☐ **async (James Gossell)**

Rules:

- Partial credit may be awarded, but you must show your work.
- You may have a single handwritten 3″ × 5″ notecard, both sides.
- Calculators are **not** allowed.
- Place a box around your $\boxed{\text{FINAL ANSWER}}$ to each question where appropriate.
- Turn off anything that might go beep during the exam.

Good luck!

Problem	Possible	Score
1	12	
2	10	
3	12	
4	11	
5	12	
6	12	
7	12	
8	9	
9	10	
Extra Credit	5	
Total	100	

1. (12 points)

Evaluate the following limits. **Show your work**, uncluding appropriate use of limit notation. If you use L'Hôpital's rule, you must indicate where you are using it by writing $\overset{H}{=}$ or $\overset{L'H}{=}$ or something similar. Use ∞ or $-\infty$ where appropriate, and if the limit does not exist, write DNE and provide a justification.

a. $\displaystyle \lim_{x\to\infty} \frac{(2x-4x^3)}{(x^3-4x^2-6)} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}} = \lim_{x\to\infty} \frac{\frac{2}{x^2}-4}{1-\frac{4}{x}-\frac{6}{x^3}} = \frac{-4}{1} = -4$

form $\frac{0}{0}$

b. $\displaystyle \lim_{t\to3} \frac{e^{t-3}-t+2}{t^2-6t+9} = \frac{e^{0}-3+2}{9-18+9} = \frac{0}{0}$

$\displaystyle \overset{h}{=} \lim_{t\to3} \frac{e^{t-3}-1}{2t-6} \overset{h}{=} \lim_{t\to3} \frac{e^{t-3}}{2} = \frac{1}{2}$

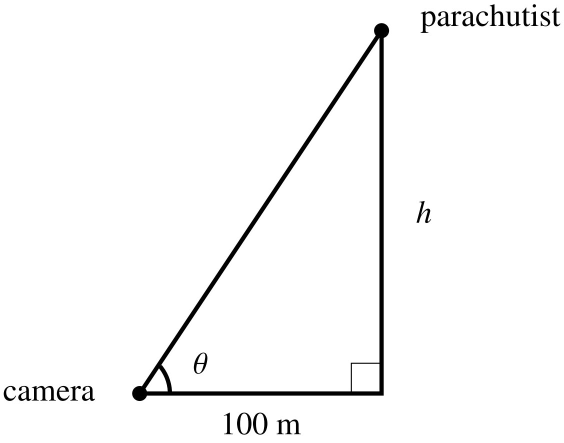
form $\frac{0}{0}$

c. $\displaystyle \lim_{\theta\to0} \frac{2\sin(\theta)-2}{1-\theta-e^{\cos(\theta)}} = \frac{2\cdot0-2}{1-0-e^{1}} = \frac{-2}{1-e} = \frac{2}{e-1}$

2. (10 points)

A camera at ground level is 100 meters from the landing site of a parachutist who is landing vertically. Let h be the height of the parachutist above the ground and let θ be the angle of elevation formed between the camera lens and the ground. (See figure.)

a. Find an equation relating h and θ.



$$\tan\theta = \frac{h}{100} \quad \text{or}$$

$$\theta = \arctan\left(\frac{1}{100}h\right)$$

b. Suppose the height of the parachutist decreases at a constant rate of 5 meters per second. At what rate does the angle θ decrease when the parachutist is 200 meters in the air? **Answer the question with a complete sentence, including units.**

$$\frac{dh}{dt} = -5, \quad \text{Find } \frac{d\theta}{dt} \text{ when } h = 200$$

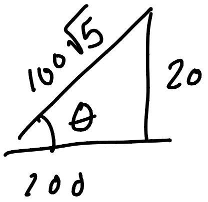
$$\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{1}{100}h\right)^2} \cdot \frac{1}{100} \cdot \frac{dh}{dt} = \frac{1}{1+2^2} \cdot \frac{1}{100} \cdot (-5) = -\frac{1}{100}$$

The angle is decreasing at a rate of $\frac{1}{100}$ rad/s.

alternate approach

$$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{100}\frac{dh}{dt},$$

when $h = 200$ m



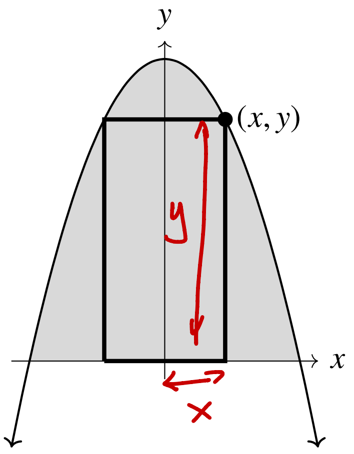
$$\sec\theta = \frac{100\sqrt{5}}{100} = \sqrt{5}$$

$$5 \cdot \frac{d\theta}{dt} = \frac{1}{100}(-5).$$

So $\frac{d\theta}{dt} = \frac{1}{100}$ rad/s

3. (12 points)

We want to determine the dimensions of the rectangle of maximum area that is inscribed between the parabola $y = 5 - x^2$ and the x-axis. Assume the base of the rectangle is on the x-axis. (See figure below; the rectangle should be inside the shaded area.)



a. Find an expression for the area A of the rectangle as a function of one variable.

$$A = 2xy = 2x(5-x^2)$$
$$= 10x - 2x^3$$

b. State the appropriate domain of the area function given the context of the problem.

$$[0, \sqrt{5}]$$

c. Use Calculus to determine where the area is maximized. Justify your conclusion with work.

Find crit. #'s: $A'(x) = 10 - 6x^2 = 0$. So $x^2 = \frac{10}{6} = \frac{5}{3}$

So $x = \sqrt{5/3}$ $\left(x = -\sqrt{\frac{5}{3}}\text{ . is not in domain}\right)$

Check crit# is a <u>max</u> : (There are 3 ways.)

① Closed-interval method

x	$A(x)$
0	0
$\sqrt{5/3}$	0
$\sqrt{5/3}$	$2\sqrt{\frac{5}{3}}\left(5-\frac{5}{3}\right)$ $= 2\sqrt{\frac{5}{3}}\left(\frac{10}{3}\right)$

② 2nd Der. test

$$A''(x) = -12x$$
$$A''\left(\sqrt{5/3}\right) = -12\left(\sqrt{\frac{5}{3}}\right) < 0$$

So A is cc down ✓

So max @ $x = \sqrt{\frac{5}{3}}$

③ 1st der. test

+++ 0 - - - -

0 $\sqrt{5/3}$ $\sqrt{5}$

$$A'(0) = 10 > 0$$
$$A'(\sqrt{5}) = 10 - 30 < 0$$

So max at $x = \sqrt{5/3}$

d. Answer the question:
The dimensions of the rectangle with largest area are width: $2\sqrt{\frac{5}{3}}$, height: $\frac{10}{3}$

$$h = 5 - x^2 = 5 - \left(\sqrt{\frac{5}{3}}\right)^2 = 5 - \frac{5}{3} = \frac{10}{3}$$

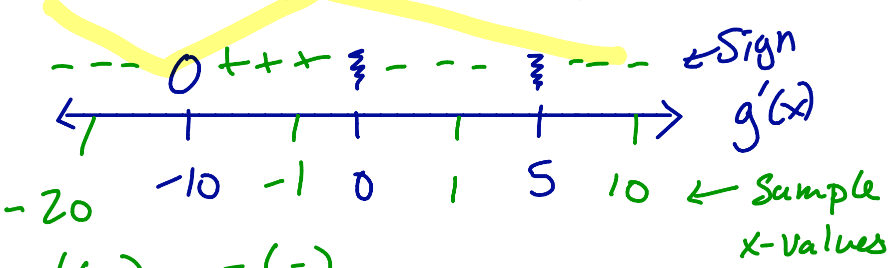
4. (11 points)

Consider the function $g(x) = \dfrac{x^{2/3}}{x-5}$. After simplification, $g'(x) = \dfrac{-(x+10)}{3x^{1/3}(x-5)^2}$.

a. What are the critical numbers of $g(x)$?

$x = -10, 0, \cancel{5}$ ← $x=5$ is not in the domain

b. At what x-values does $g(x)$ have local maximum(s)? At what x-values does g(x) have local minimum(s)? Clearly show work to **justify** your answers.



$g'(-20) = \dfrac{-(-)}{(-)(+)} = -$

$g'(1) = \dfrac{-(+)}{+(+)} = -$; $g'(10) = \dfrac{-(+)}{+(+)} = -$

$g'(-1) = \dfrac{-(+)}{-(+)} = +$

Local maximum(s): $x =$ 0 Local minimum(s): $x =$ -10

(If none, write "none".)

c. Does $g(x)$ have any horizonal asympotes? If it does, write the equations of any horizontal asymptote(s) of $g(x)$, and justify each answer by writing a limit. If it doesn't, explain why $g(x)$ does not have any horizontal asymptotes and write "none".

$$\lim_{x\to+\infty} \frac{x^{2/3}}{x-5} = 0, \quad \lim_{x\to-\infty} \frac{x^{2/3}}{x-5} = 0$$

Horizontal asymptote equation(s): $y=0$

(If none, write "none".)

5. (12 points)

Sketch a graph of a function $f(x)$ that satisfies all of the following properties.

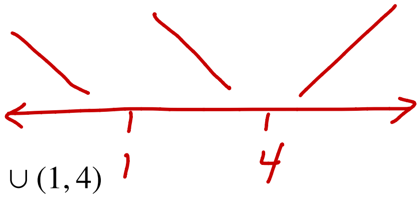
After drawing the graph:

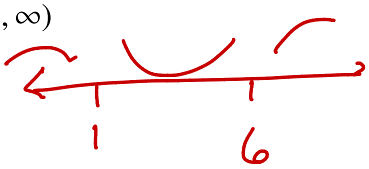
- **Label** on the graph the following things, if they exist, by drawing a point on the graph and labeling: any local maximums by writing LOCAL MAX, local minimums by writing LOCAL MIN, inflection points by writing IP
- Draw any horizontal and vertical asymptotes with dashed lines and **label** them with their equation.
- Mark any important x-values and y-values on the x- and y-axes.

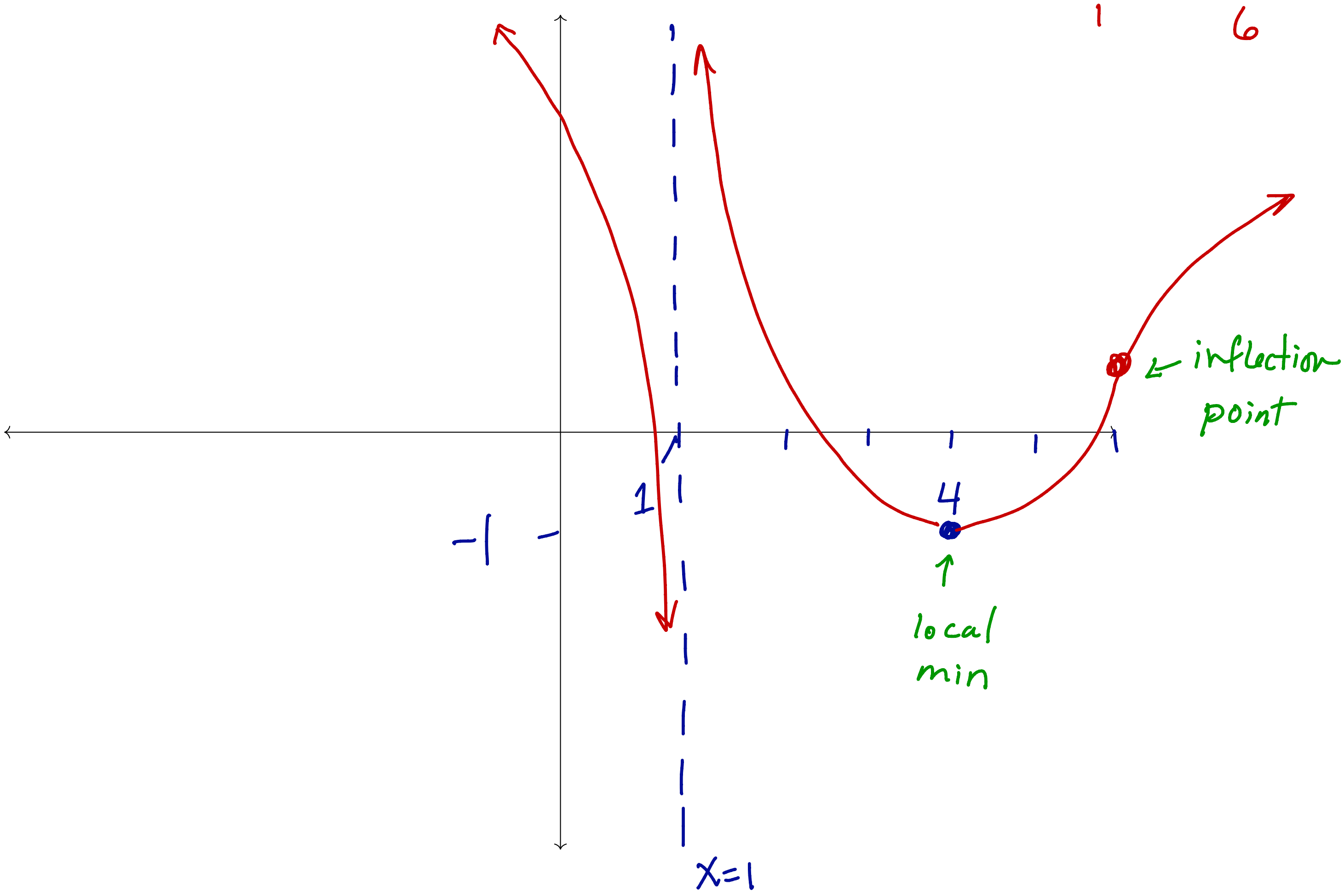
Properties:

- Domain is $(-\infty, 1) \cup (1, \infty)$
- $f(4) = -1$ and $f'(4) = 0$
- $\lim_{x \to 1^+} f(x) = \infty$

- $f'(x) < 0$ on $(-\infty, 1) \cup (1, 4)$
- $f'(x) > 0$ on $(4, \infty)$
- $f''(x) < 0$ on $(-\infty, 1) \cup (6, \infty)$
- $f''(x) > 0$ on $(1, 6)$

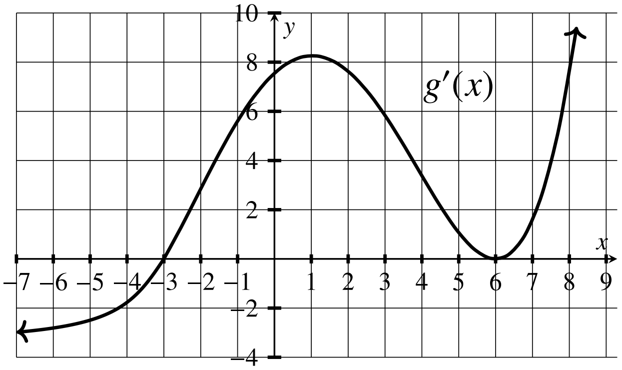






6. (12 points)

The graph shown below is the graph of the **derivative** $g'(x)$ of a function $g(x)$. Answer the following questions about the **original** function $g(x)$.



a. Determine the critical numbers of $g(x)$. (Notice that $g(x)$ is **not** shown on the graph!)

$x = -3, 6$

b. Determine the intervals where g is **increasing** and where g is **decreasing**. If none write "none".

Increasing: $(-3, \infty)$ Decreasing: $(-\infty, -3)$

c. Fill in the blanks (if none, write "none"):

$g(x)$ has (a) local maximum(s) at $x =$ none and (a) local minimum(s) at $x =$ -3.

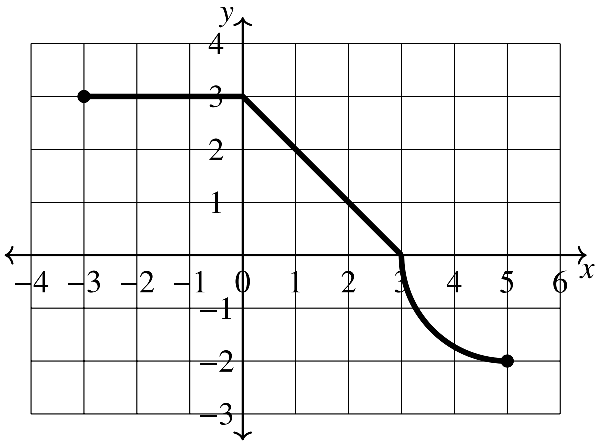
d. Find all intervals where g is **concave up** and where g is **concave down**. (If none write "none".)

Concave up: $(-\infty, 1) \cup (6, \infty)$ Concave down: $(1, 6)$

e. Fill in the blanks: $g(x)$ has (an) inflection point(s) at $x =$ $1, 6$. (If none, write "none".)

7. (12 points)

The graph of the function $G(x)$ is below. The function $G(x)$ has domain $[-3,5]$ and the portion of the graph on the interval $[3,5]$ is one quarter of a circle of radius 2 centered at the point $(5,0)$.



a. Determine $\int_{-2}^{3} G(x)\,dx$. $= 6 + 4.5 = 10.5$

b. Determine $\int_{-3}^{5} G(x)\,dx$. $= 9 + 4.5 - \frac{1}{4}\pi(2)^2 = 13.5 - \pi$

c. Determine $\int_{-3}^{5} 2G(x) + 3\,dx$ (Hint: Use part (b) above.)

$= 2(13.5 - \pi) + 3(8)$

$= 27 - 2\pi + 24$

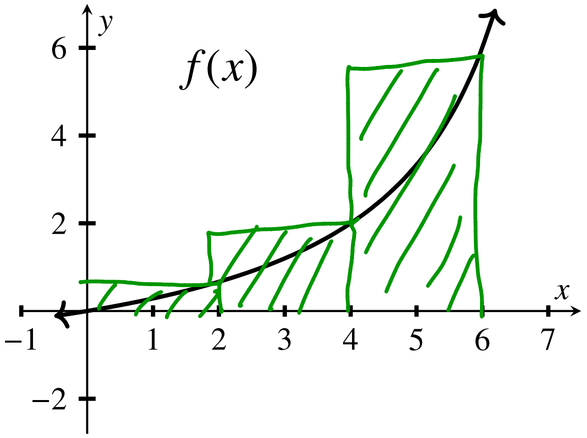
$= 51 - 2\pi$

8. (9 points)

Consider the function

$$f(x) = \frac{2x}{8 - x}.$$

A portion of the graph of this function is shown to the right.



a. Compute R_3 on the interval $[0, 6]$. That is, approximate $\int_0^6 f(x)\,dx$ using 3 right-hand rectangles. **Draw the rectangles on the graph.**

$$R_3 = 2\left(f(2)+f(4)+f(6)\right) = 2\left(\frac{4}{6}+\frac{8}{4}+\frac{12}{2}\right) = 2\left(\frac{2}{3}+4+6\right) = 11\tfrac{1}{3} = \frac{34}{3}$$

b. List **two distinct strategies** to compute a more accurate approximation of $\int_0^6 f(x)\,dx$.

- more rectangles
- midpoints

9. (10 points)

Evaluate the indefinite integrals below. (Give the most generic answer.)

a. $\int \left(3x^3 + \cos(x) - e^x + \sqrt{5}\right)\,dx = \frac{3}{4}x^4 + \sin(x) - e^x + \sqrt{5}x + C$

b. $\int \frac{1 + x^{\frac{1}{3}} + x^4}{x}\,dx = \int\left(x^{-1} + x^{-2/3} + x^3\right)dx$

$$= \ln|x| + 3x^{1/3} + \frac{1}{4}x^4 + c$$

Extra Credit (5 points)

A population of bacteria can be modeled by the function $P(t) = t^{k/t}$, where t is time, measured in hours, P is the number of bacteria, measured in thousands, and k is a fixed positive constant.

a. Compute $\lim\limits_{t\to\infty} P(t)$.

form ∞^0

$$\lim_{t\to\infty} t^{k/t} = \boxed{e^0} = 1$$

take natural log of P(t).

$$\lim_{t\to\infty} \frac{k\ln t}{t} \overset{\textcircled{h}}{=} \lim_{t\to\infty} \frac{k\cdot\frac{1}{t}}{1} = \frac{0}{1} = 0$$

form $\frac{\infty}{\infty}$

b. Interpret this limit by writing a complete sentence, including units, using the context of the model.

In the long run, the population of bacteria stabilizes at 1 thousand bacteria.