Your Name

Solutions

Your Signature

Instructor Name

Problem	Total Points	Score
1	16	
2	12	
3	6	
4	6	
5	8	
6	10	
7	12	
8	6	
9	10	
10	8	
11	6	
Total	100	

- This test is closed notes and closed book.
- You may **not** use a calculator.
- In order to receive full credit, you must **show your work.** Be wary of doing computations in your head. Instead, write out your computations on the exam paper.
- **PLACE A BOX AROUND** $\boxed{\textbf{YOUR FINAL ANSWER}}$ **to each question** where appropriate.
- If you need more room, use the backs of the pages and indicate to the reader that you have done so.
- Raise your hand if you have a question.

1. (16 points) Find the derivative for each of the following functions. (For parts (a)-(c), you do not need to simplify your answers.)

(a) $h(x) = \dfrac{\arctan(x)}{1+x}$

$$h'(x) = \frac{(1+x)\left(\frac{1}{1+x^2}\right) - 1\cdot\arctan(x)}{(1+x)^2} = \boxed{\frac{(1+x)-(1+x^2)\arctan(x)}{(1+x)^2(1+x^2)}}$$

(b) $f(x) = \sec\left(\sqrt{1-x^2}\right)$ $= \sec\left((1-x^2)^{1/2}\right)$

$$f'(x) = \sec\left((1-x^2)^{1/2}\right)\tan\left((1-x^2)^{1/2}\right)\left(\tfrac{1}{2}(1-x^2)^{-1/2}\right)(-2x)$$

$$= \boxed{\frac{-x\sec\left(\sqrt{1-x^2}\right)\tan\left(\sqrt{1-x^2}\right)}{\sqrt{1-x^2}}}$$

(c) $y = \dfrac{3}{x} + 3\ln(x) - \tan(3\pi)$ $= 3x^{-1} + 3\ln(x) - \tan(3\pi)$

$$\boxed{y' = -3x^{-2} + \frac{3}{x}}$$

(d) $2x^2 - 5xy + 4y^2 = 2$ (Solve for dy/dx.) (Implicit differentiation.)

$$4x - 5\cdot y - 5x\cdot\frac{dy}{dx} + 8y\frac{dy}{dx} = 0$$

$$(8y-5x)\left(\frac{dy}{dx}\right) = 5y - 4x$$

$$\boxed{\frac{dy}{dx} = \frac{5y-4x}{8y-5x}}$$

2. (12 points) Evaluate the following integrals.

(a) $g(x) = \int \left(\frac{2}{x} + 2x^{1/3} - e^2\right) \mathrm{d}x$ $= 2\ln(|x|) + 2 \cdot \frac{3}{4} x^{4/3} - e^2 x + C$

$= \boxed{2\ln(|x|) + \frac{3}{2} x^{4/3} - e^2 x + C}$

(b) $h(x) = \int 4\cos^3(x)\sin(x)\mathrm{d}x$ $= -4\int u^3 du = -u^4 + C$

let $u = \cos x$

$du = -\sin x \, dx$

$= \boxed{-(\cos x)^4 + C}$

(c) $f(x) = \int \left(x\sqrt{2x-1}\right) \mathrm{d}x$ $= \frac{1}{2}\int \left(\frac{u+1}{2}\right) u^{\frac{1}{2}} du = \frac{1}{4}\int (u+1) u^{\frac{1}{2}} du$

let $u = 2x-1$

$du = 2dx$

$\frac{1}{2} du = dx$

$x = \frac{u+1}{2}$

$= \frac{1}{4}\int \left(u^{\frac{3}{2}} + u^{\frac{1}{2}}\right) du = \frac{1}{4}\left[\frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}}\right] + C$

$= \boxed{\frac{1}{10}(2x-1)^{5/2} + \frac{1}{6}(2x-1)^{3/2} + C}$

3. (6 points) Let $f(x) = \frac{1}{x}$.

(a) Find the average rate of change of f from $x = 1$ to $x = 3$. Simplify your answer if possible.

$$\text{avg. r.o.c.} = \frac{f(3)-f(1)}{3-1} = \frac{1}{2}\left(\frac{1}{3}-\frac{1}{1}\right) = \frac{1}{2}\left(\frac{-2}{3}\right) = \boxed{\frac{-1}{3}}$$

(b) Find $f'(x)$ using the definition of the derivative.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{x+h}-\frac{1}{x}\right) = \lim_{h\to 0} \frac{1}{h}\left(\frac{x-(x+h)}{(x+h)(x)}\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{-h}{(x+h)(x)}\right) = \lim_{h\to 0} \frac{-1}{(x+h)(x)} = \boxed{\frac{-1}{x^2}}$$

4. (6 points) Let $f(x) = x^{2/3}$.

(a) Find the linearization $L(x)$ of $f(x)$ at $x = 8$.

$f(x) = x^{2/3}$, $f(8) = 8^{2/3} = 4 = y_0$

$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}}$; $f'(8) = \frac{2}{3(8)^{1/3}} = \frac{2}{3\cdot 2} = \frac{1}{3} = m$

$y - 4 = \frac{1}{3}(x-8)$ or $\boxed{L(x) = 4 + \frac{1}{3}(x-8)}$

(b) Use your answer in part a to estimate $(8.1)^{2/3}$. Write your answer as a common fraction.

$$(8.1)^{2/3} = f(8.1) \approx L(8.1) = 4 + \frac{1}{3}(8.1-8) = 4 + \frac{1}{3}\left(\frac{1}{10}\right)$$

$$= 4 + \frac{1}{30} = \boxed{\frac{121}{30}}$$

5. (8 points) The height, h, of water in a ditch is given by

$$h(t) = \frac{2 + \sin(\pi t)}{1 + t},$$

where h is measured in feet and t is measured in days.

(a) Find and **interpret** $h(3)$ in the context of the problem. (Your expression for $h(3)$ should be simplified.)

$$h(3) = \frac{2+\sin(3\pi)}{1+3} = \frac{2}{4} = \frac{1}{2} \text{ feet}$$

After 3 days have passed, there is $\frac{1}{2}$ foot of water in the ditch.

(b) Find $h'(t)$. (You do not need to simplify your answer.)

$$h'(t) = \frac{(1+t)(\cos(\pi t))(\pi) - (1)(2+\sin(\pi t))}{(1+t)^2}$$

(c) Interpret $h'(3) \approx -0.91$ in the context of the problem.

After 3 days, the height of water in the ditch is decreasing at a rate of 0.91 ft per day.

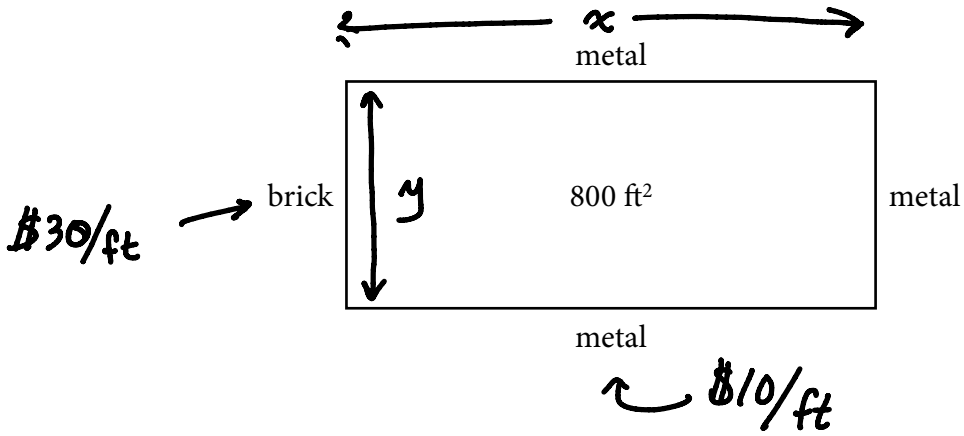
(d) Find and **interpret** $\lim_{t\to\infty} h(t)$. (Hint: $-1 \le \sin(x) \le 1$.)

$$\boxed{\lim_{t\to\infty} \frac{2+\sin(t)}{1+t} = 0}$$ since $\frac{1}{1+t} \le \frac{2+\sin(t)}{1+t} < \frac{3}{1+t}$ and $\lim_{t\to\infty} \frac{1}{1+t} = 0$

and $\lim_{t\to\infty} \frac{3}{1+t} = 0$.

This limit tells us that, eventually, the ditch has no water.

6. (10 points) A landscape architect wishes to enclose a rectangular garden on one side by a brick wall costing \$30 per foot and on the other three sides with a metal fence costing \$10 per foot. The area of the garden is to be 800ft^2. What are the dimensions of the garden that minimize the cost of the fencing? (For full credit, you must justify your answer.)



area $= A = xy = 800$
or
$y = 800x^{-1}$

goal: minimize cost

Cost $= C = 30y + 10y + 2(10)(x) = 40y + 20x$.

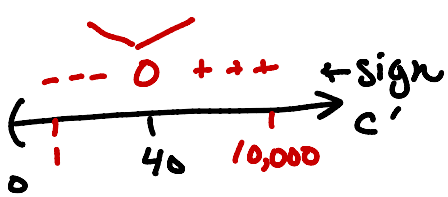
So $C(x) = 40(800x^{-1}) + 20x = 32000x^{-1} + 20x$.

Now $C'(x) = -32000x^{-2} + 20x = 0$

$20x = \frac{32000}{x^2}$ or $x^3 = \frac{32000}{20} = 1600$

So $x = 40$.

First Der. Test:



$C(x)$ has a local min at $x = 40$.

Is $x = 40$ a global min?

option 1: Yes. Because $x = 40$ is the only crit. point in the domain in which $C(x)$ is continuous.

option 2: Yes. Because $C''(x) = 64000x^{-3} + 20$ which is always positive on in the domain. So $C(x)$ is c cup.

answer the question

dimensions: $x = 40$ ft
$y = 20$ ft.

7. (12 points) Let $g(x) = \dfrac{e^x}{1+x}$. Note first and second derivatives are

$$g'(x) = \frac{xe^x}{(1+x)^2} \quad \text{and} \quad g''(x) = \frac{e^x(x^2+1)}{(1+x)^3}. \leftarrow \text{always}+$$

(a) Evaluate the following limits.

i. $\lim_{x\to\infty} g(x) = \lim_{x\to\infty} \frac{e^x}{1+x} \overset{H}{=} \lim_{x\to\infty} \frac{e^x}{1} = \infty$

ii. $\lim_{x\to-\infty} g(x) = \lim_{x\to\infty} \frac{e^{-x}}{1-x} = \lim_{x\to\infty} \frac{1}{e^x(1-x)} = 0$ since $e^x(1-x) \to -\infty$.

iii. $\lim_{x\to-1^-} g(x) = \lim_{x\to-1^-} \frac{e^x}{1+x} = -\infty$ since $e^x \to \frac{1}{e}$ and $1-x \to 0^-$.

(b) Sketch the graph of $g(x)$. Label any asymptotes, x- and y-intercepts, local minimums and local maximums, and inflection points, if appropriate.

Note (ii) and (iii) imply $y=0$ is a horizontal asymptote and $x=-1$ is a vertical asymptote.

$g'(x) = 0$ when $x=0$

$- \mid - \; 0 \; +$ ← sign g'

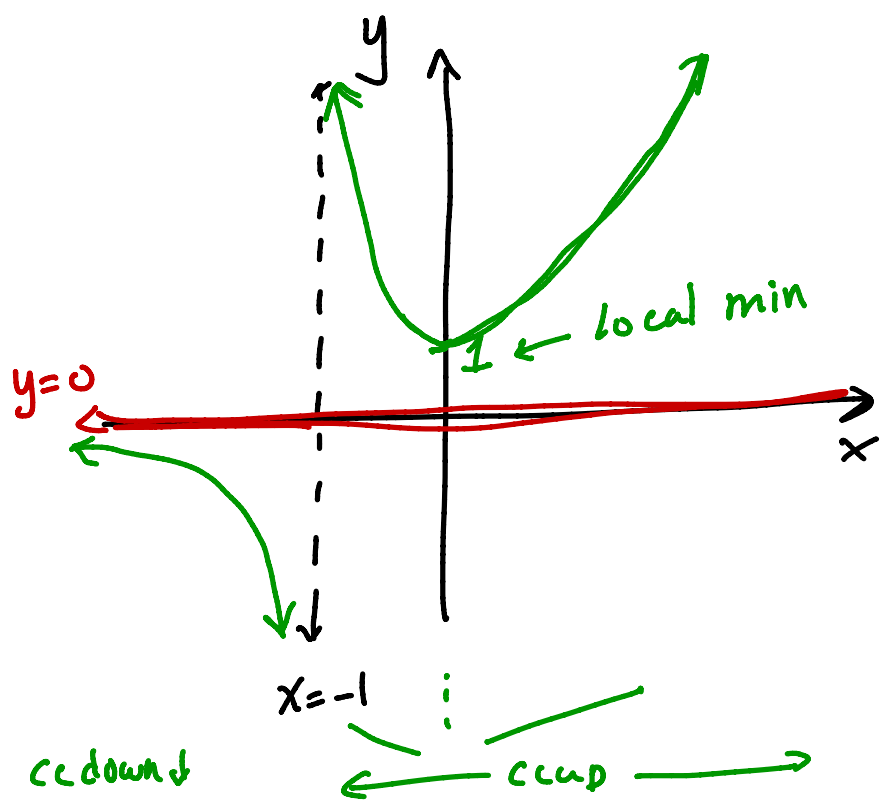
$-1 \quad 0$

$g'' \neq 0$.

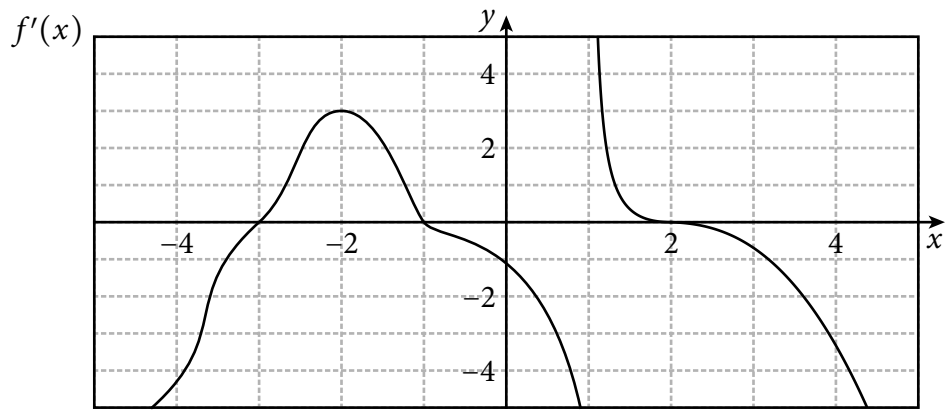
$- - - \mid + + +$ ← sign g''

-1

ccdown | ccup



8. (6 points) The graph of **the derivative** of $f(x)$, $f'(x)$, is shown below. Questions (a) through (d) concern the function $f(x)$.



(a) For what intervals(s) is $f(x)$ increasing?

where $f' > 0$.

Ans: $(-3,-1) \cup (1,2)$

(b) For what intervals(s) is $f(x)$ concave up?

where f' is increasing

Ans: $(-\infty,-2)$

(c) What value(s) of x give $f(x)$ a relative maximum?

$x=-1$, $x=2$

where f' changes from + to −

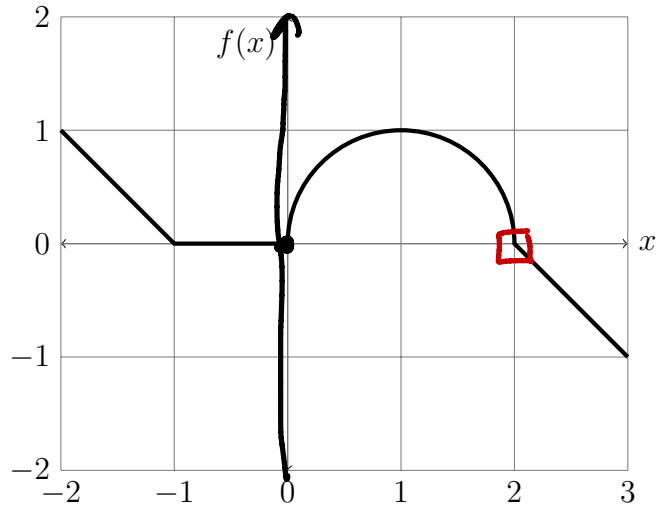
(d) What value(s) of x give $f(x)$ inflection points?

where f' changes from increasing to decreasing.

$x=-2$

9. (10 points) The function $f(x)$ has been graphed below. The curve for $0 < x < 2$ is an upper half circle. Define a new function $g(x)$, as

$$g(x) = \int_0^x f(s)\, ds.$$



Use the graph above to answer the questions below.
Note: Pay attention to whether question concerns the function f, f', g or g'.

(a) What is the value of $f(0)$?

$f(0) = 0$

(b) What is the value of $g(3)$?

signed area under curve from $x=0$ to $x=3$

Ans: $\frac{1}{2}\pi(1)^2 - \frac{1}{2}(1)(1)$
$= \frac{1}{2}\pi - \frac{1}{2} = \frac{1}{2}(\pi - 1)$

(c) What is the value of $g(-2)$?

$g(-2) = \int_0^{-2} f(s)ds = -\int_{-2}^{0} f(s)ds = -\frac{1}{2}$ area under curve.

(d) What is the value of $f'(2)$?

DNE.
A corner at $x=2$. So $f'(2)$ is undefined.

(e) What is the value of $g'(1)$?

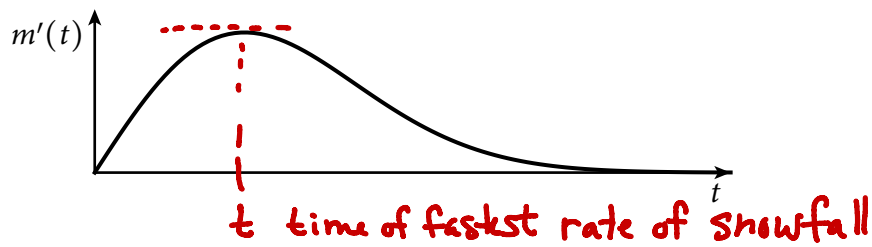
$g'(x) = f(x)$ by FTC part I.

Ans: $g'(1) = f(1) = 1$

10. (8 points) Snow is accumulating on my deck. The total amount of snow on my deck is $m(t)$ kilograms, where $t > 0$ is measured in hours. The instantaneous rate of accumulation is

$$m'(t) = 4te^{-t^2}$$

kilograms per hour.



(a) At what time is the **rate** of snow accumulation at its peak?

want: $m''(t) = 0$.

$$m''(t) = 4\left[1 \cdot e^{-t^2} + t \cdot \left(e^{-t^2}\right)(-2t)\right] = 4e^{-t^2}\left[1 - 2t^2\right] = 0$$

$t^2 = \frac{1}{2}$ only positive t makes sense.

$t = \pm\sqrt{1/2}$ answer: $t = 1/\sqrt{2}$ hr.

(b) In the diagram above, label the time, t, obtained in part (a). (in red)

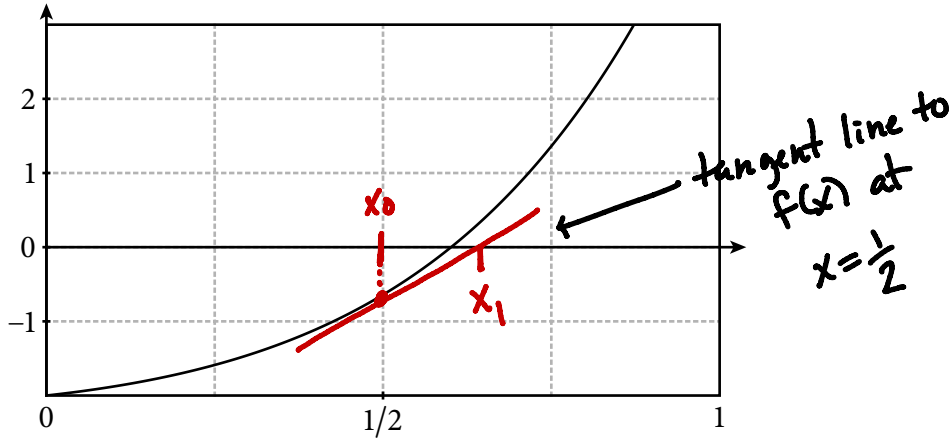
(c) Assume that at time $t = 0$ there are 10kg of snow on the deck. How much snow is on the deck at time $t = 2$ hours?

total amount of snow = snow present + snow accumulated

$$= 10 + \int_0^2 4te^{-t^2}\,dt = 10 + \left[-2e^{-t^2}\right]_0^2$$

$$= 10 + \left[-2e^{-4} - (-2e^0)\right] = 10 + 2\left[1 - \frac{1}{e^4}\right] \text{ kg}$$

11. (6 points) Consider the function $f(x) = xe^{2x} - 2$ graphed below.



(a) Approximate the solution of $f(x) = 0$ using **JUST ONE** iteration of Newton's method starting from an initial guess of $x_0 = 1/2$ to compute a new estimate: x_1. It is OK to leave your answer unsimplified, but your answer should be an expression you could compute if you had a calculator.

$$f'(x) = 1 \cdot e^{2x} + x \cdot e^{2x} \cdot 2$$

$$x_0 = \frac{1}{2}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Ans

$$x_1 = \frac{1}{2} - \frac{\frac{1}{2}e^1 - 2}{e^1 + \frac{1}{2}e^1 \cdot 2}$$

$$= \frac{1}{2} - \frac{\frac{1}{2}e - 2}{2e} = \frac{1}{2} - \left(\frac{e-4}{4e}\right)$$

(b) In the figure above, indicate the point x_1 you computed in part (b) and demonstrate in the diagram how x_1 was obtained from x_0.

In red