Your Name

Solutions

Your Signature

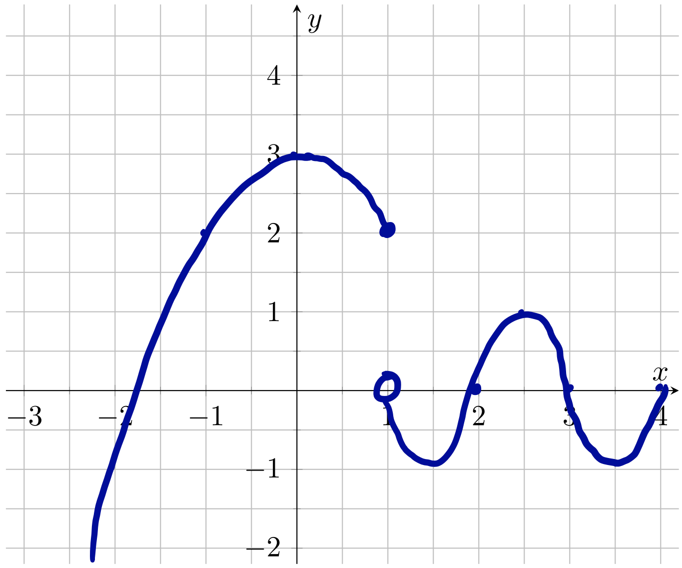
Instructor Name

Problem	Total Points	Score
1	20	
2	21	
3	20	
4	12	
5	15	
6	12	
Extra Credit	(5)	
Total	100	

- This test is closed notes and closed book.
- You may **not** use a calculator.
- In order to receive full credit, you must **show your work.** Be wary of doing computations in your head. Instead, write out your computations on the exam paper.
- **PLACE A BOX AROUND** $\boxed{\textbf{YOUR FINAL ANSWER}}$ **to each question** where appropriate.
- If you need more room, use the backs of the pages and indicate to the reader that you have done so.
- Raise your hand if you have a question.

1. Given the function $f(x) = \begin{cases} 3 - x^2 & x \leq 1 \\ \sin(\pi x) & x > 1 \end{cases}$

(a) (8 points) Sketch the graph of $f(x)$



(b) (2 points each) Evaluate each of the limits below.

i. Find $\lim_{x \to 1^+} f(x)$ $\boxed{0}$

ii. Find $\lim_{x \to 1^-} f(x)$ $\boxed{2}$

iii. Find $\lim_{x \to \infty} f(x)$ $\boxed{\text{DNE}}$

iv. Find $\lim_{x \to -\infty} f(x)$ $\boxed{-\infty}$

(c) (4 points) Determine whether the function is continuous at $x = 1$ and justify your answer using the definition of continuity.

The function is <u>not</u> continuous:

$\lim_{x \to 1} f(x)$ does not exist, since $\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$.

2. (21 points) Evaluate the following limits and justify your answers.

(a) $\displaystyle\lim_{x\to -1} \frac{x^2+4x+3}{x^2+x} = \lim_{x\to -1} \frac{(x+1)(x+3)}{x(x+1)}$

$$= \lim_{x\to -1} \frac{x+3}{x}$$

$$= \frac{-1+3}{-1}$$

$$= \boxed{-2}$$

(b) $\displaystyle\lim_{x\to 5^+} \frac{x^2+2}{5-x}$

As $x \to 5^+$, $5-x \to 0^-$ and $x^2+2 \to 27$.

So $\displaystyle\lim_{x\to 5^+} \frac{x^2+5}{5-x} = \frac{27}{0^-} = \boxed{-\infty.}$

(c) $\displaystyle\lim_{x\to\infty} \frac{3-5e^x}{2e^x+1} = \lim_{x\to\infty} \frac{e^{-x}}{e^{-x}} \frac{3-5e^x}{2e^x+1}$

$$= \lim_{x\to\infty} \frac{3e^{-x}-5}{2+e^{-x}}$$

$$= \frac{0-5}{2+0} = \boxed{-\frac{5}{2}}$$

(d) $\displaystyle\lim_{x\to -17} |x+15|$

Since $f(x)=|x|$ is continuous,

$$\lim_{x\to -17} |x+15| = |-17+15| = |-2| = \boxed{2}$$

3. (20 points) A population of moose is declining. The population at time t is

$$P(t) = \frac{1000}{1+t}$$

where P is the number of moose and t is measured in years.

(a) (5 points) Compute the average rate of change of the population from time $t = 1$ to $t = 4$ years.

rate of change: $\dfrac{P(4)-P(1)}{4-1} = \dfrac{\frac{1000}{5} - \frac{1000}{2}}{3}$

$= \dfrac{200-500}{3}$

$= -100$ moose/year

(b) (5 points) Compute the average rate of change of the population from time $t = 1$ to $t = b$ years.

rate of change: $\dfrac{P(b)-P(1)}{b-1} = \dfrac{\frac{1000}{1+b} - \frac{1000}{2}}{b-1}$

$= \dfrac{\frac{2000 - 1000 - b \cdot 1000}{2(b+1)}}{b-1}$

$= \dfrac{\frac{1000(1-b)}{2(b+1)}}{b-1}$

$= \dfrac{1000}{2(b+1)}$

(part (c) continued on next page ⟶)

(c) (10 points) Using a limit, find the instantaneous rate of change of the moose population at time $t = 1$ years. No credit will be given for using a derivative rule you may have learned in the past. You must compute the rate of change from the limit definition.

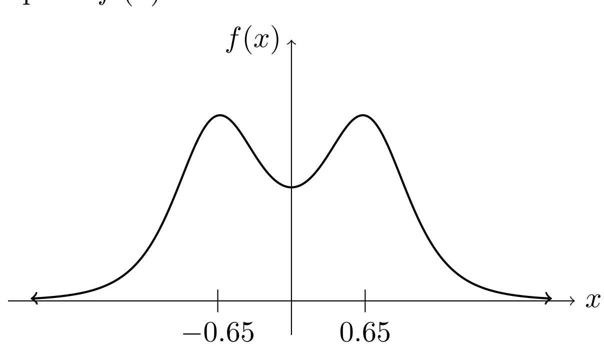
$$P'(1) = \lim_{b \to 1} \frac{P(b) - P(1)}{b-1} = \lim_{b \to 1} \frac{\frac{1000}{1+b} - \frac{1000}{2}}{b-1}$$

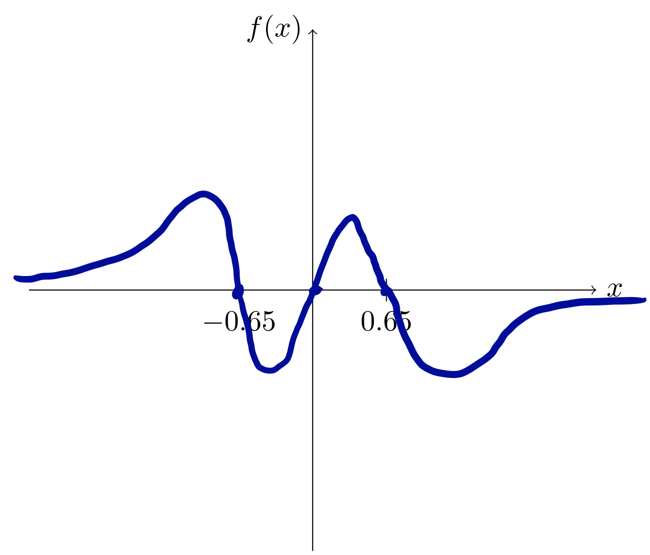
$$= \lim_{b \to 1} \frac{2000 - 1000 - 1000b}{2(1+b)(b-1)}$$

$$= \lim_{b \to 1} \frac{1000(1-b)}{2(1+b)(b-1)}$$

$$= \lim_{b \to 1} \frac{-1000}{2(1+b)} = -\frac{1000}{4} = -250 \ \frac{\text{moose}}{\text{year}}$$

4. (12 points) The axis on the left is the graph of a function $f(x)$. In the axis on the right, sketch the graph of $f'(x)$.





5. (15 points) A cup of coffee is cooling on a desk. The temperature of the coffee is

$$T(t) = 20 + 80 \cdot 10^{-t/30}.$$

where t is measured in minutes from some initial time and T is measured in degrees Celsius.

(a) At what time t will the coffee's temperature be equal to 60°C?

We solve $T(t) = 60$:

$$20 + 80 \cdot 10^{-t/30} = 60$$

$$80 \cdot 10^{-t/30} = 40$$

$$10^{-t/30} = 1/2$$

$$-\frac{t}{30} = \log_{10}(1/2)$$

So: $t = -30 \log_{10}(1/2)$

$= \boxed{30 \log_{10}(2)}$

(b) In the context of the problem, interpret $T'(t)$. (Include units.)

$T'(t)$ is the instantaneous rate of change of the temperature of the coffee at time t.

Units: °C/minute

(c) What does the statement $T'(10) = -2.85$ mean?

At $t = 10$ minutes, the coffee is cooling at an instantaneous rate of 2.85 °C/minute.

6. (12 points) Assume $f(x) = \frac{6x}{x+2}$ and $f'(x) = \frac{12}{(x+2)^2}$. Find the equation of the line tangent to the graph $f(x)$ when $x = 2$.

Slope at $x=2$: $\frac{12}{4^2} = \frac{12}{16} = \frac{3}{4}$

point: $x=2$, $y = f(2) = \frac{12}{6} = 2$

$y-2 = \frac{3}{4}(x-2)$ ← point-slope form

$$\boxed{y = 2 + \frac{3}{4}(x-2)}$$

Extra Credit (5 points) Use the Intermediate Value Theorem to show that the two curves $f(x) = 4x + \sin x$ and $g(x) = x - e^{-x}$ must intersect.

Note: $f(x) = g(x) \iff f(x) - g(x) = 0$.

Let $h(x) = f(x) - g(x) = 4x + \sin(x) - x + e^{-x}$

$= 3x + \sin(x) + e^{-x}$

Observe: $h(0) = 3\cdot 0 + \sin(0) + e^{0} = 1 > 0$

$h(-1) = -3 + \sin(-1) + e^{1} < -3 + 0 + e < 0$

Since h is continuous, there is a solution of

$h(x) = 0$ for some $x \in [-1, 0]$.