Your Name: Solutions

Your Signature

Instructor Name

Problem	Total Points	Score
1	15	
2	15	
3	10	
4	20	
5	15	
6	15	
7	10	
Extra Credit	(5)	
Total	100	

- This test is closed notes and closed book.
- You may **not** use a calculator.
- In order to receive full credit, you must **show your work.** Be wary of doing computations in your head. Instead, write out your computations on the exam paper.
- **PLACE A BOX AROUND YOUR FINAL ANSWER to each question** where appropriate.
- If you need more room, use the backs of the pages and indicate to the reader that you have done so.
- Raise your hand if you have a question.

Geometric Formulas

sphere
$V = \frac{4}{3}\pi r^3$
$A = 4\pi r^2$

cylinder
$V = \pi r^2 h$

cone
$V = \frac{1}{3}\pi r^2 h$
$A = \pi r\sqrt{r^2 + h^2}$

1. (15 points) Consider the function $f(x) = \dfrac{(x-1)^3}{x^2}$. We have computed for you ← Hey! domain $\mathbb{R}$ except $\underline{\underline{x=0}}$.

$$f'(x) = \frac{(x-1)^2(x+2)}{x^3} \quad \text{and} \quad f''(x) = \frac{6x-6}{x^4}.$$

For full credit, show your work.

(a) Find the intervals where $f(x)$ is increasing and decreasing.

$f' = 0$ when $x = 1$ and $x = -2$

f' undefined when $x = 0$

+ 0 − ≀ + 0 + ← sign f'

−3 −2 −1 0 ½ 1 10

$\boxed{f \text{ is increasing on } (-\infty,-2)\cup(0,\infty) \text{ and decreasing on } (-2,0)}$

(b) Find the intervals where $f(x)$ is concave up and concave down.

$f'' = 0$ when $x = 1$

f'' undf. when $x = 0$

− ≀ − 0 + ← sign f''

−1 0 ½ 1 10

$\boxed{f \text{ is ccup on } (1,\infty) \text{ and ccdown on } (-\infty,0)\cup(0,1)}$

(c) Classify all critical points of $f(x)$.

$\boxed{x=1 : \text{neither.}}$ f' doesn't change signs on either side of $x=1$.

$\boxed{x=-2 : \text{local max.}}$ $f''(-2) < 0$.

(Note $x=0$ NOT a critical point because it isn't in the domain of $f(x)$.)

2. (15 points) Evaluate the following limits. [Note: You should be careful to apply L'Hospital's rule *only* when appropriate.]

(a) $\lim_{x\to 0} \frac{\ln(x+1)}{1-e^{5x}} \overset{H}{=} \lim_{x\to 0} \frac{\frac{1}{x+1}}{-5e^{5x}} = \frac{\frac{1}{1}}{-5} = \boxed{\frac{-1}{5}}$

form $\frac{0}{0}$

(b) $\lim_{x\to 0^+} \sqrt{x}\ln x = \lim_{x\to 0^+} \frac{\ln x}{x^{-1/2}} \overset{H}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{2}x^{-3/2}} = \lim_{x\to 0^+} \frac{1}{x}\cdot\frac{-2}{1}\cdot x^{3/2}$

form $0\cdot -\infty$

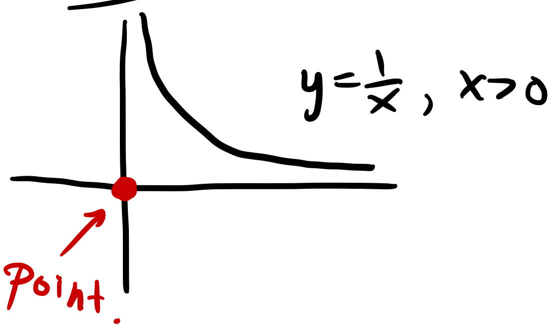
form $\frac{-\infty}{\infty}$

$= \lim_{x\to 0^+} -2x^{1/2} = \boxed{0}$

(c) $\lim_{\theta\to\pi} \frac{\sin\theta}{1-\cos\theta} = \frac{\sin\pi}{1-\cos(\pi)} = \frac{0}{2} = \boxed{0}$

3. (10 points) Show that the point on the curve $y = \frac{1}{x}$ with $x > 0$ that is closest to the point $(0,0)$ is the point $(1,1)$. For full credit, you must provide an argument showing that an absolute minimum is attained at the stated point.

Picture



goal: minimize distance (squared.)

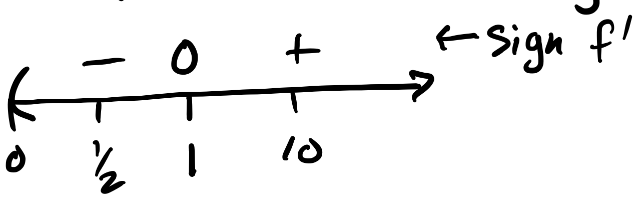
$$D = (x-0)^2 + (y-0)^2 = x^2 + y^2$$

$$D(x) = x^2 + \left(\frac{1}{x}\right)^2 = x^2 + x^{-2}$$

domain $(0, \infty)$.

$$D'(x) = 2x - 2x^{-3} = 0; \quad \text{So} \quad x - \frac{1}{x^3} = 0 \text{ or } x^4 - 1 = 0.$$

crit. pts: $x = \pm 1$. Only $x=1$ is in our domain.



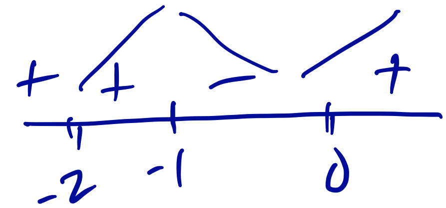
① First Derivative Test implies D has a local min @ $x=1$. It has an absolute minimum here because f is decreasing on the left of 1 and increasing on the right for every x-value in the domain.

OR

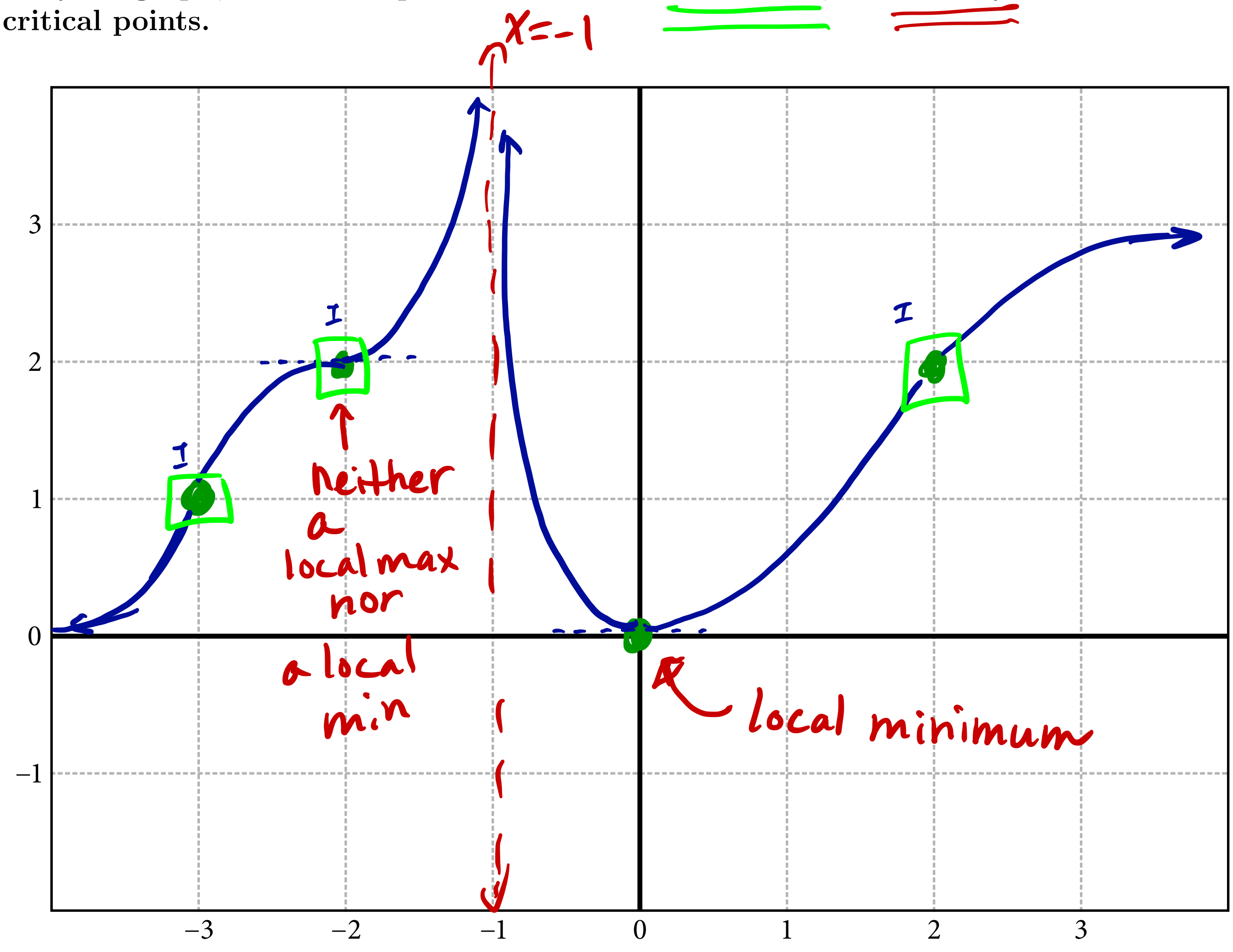
② $D''(x) = 2 + 6x^{-2} > 0$ for all x.
So D is concave up everywhere on $(0, \infty)$.
So a local minimum is an absolute minimum.

4. (20 points) Sketch the graph of a function $f(x)$ that has the following properties.

- $f(x)$ is defined for all x except $x = -1$. ✓
- $f(-3) = 1$, $f(-2) = 2$, $f(0) = 0$, and $f(2) = 2$. ✓
- $f(x)$ has a vertical asymptote at $x = -1$. ✓
- $\lim_{x\to-\infty} f(x) = 0$ and $\lim_{x\to\infty} f(x) = 3$. ✓
- $f'(x)$ is zero at $x = -2$ and $x = 0$, is positive for $x < -2$, $-2 < x < -1$ and $x > 0$, and is negative elsewhere.
- $f''(x)$ is zero at $x = -3$, $x = -2$ and $x = 2$, is positive for $x < -3$, $-2 < x < -1$ and $-1 < x < 2$, and is negative elsewhere.



On your graph, mark each point of inflection with a box, and classify all of the critical points.



5. (15 points)

(a) Find the linearization of $f(x) = \sin(x)$ at $x = \pi/4$.

$f(\pi/4) = \sin(\pi/4) = \sqrt{2}/2$

$f'(x) = \cos x$

$f'(\pi/4) = \cos(\pi/4) = \sqrt{2}/2$

Point: $\left(\frac{\pi}{4}, \sqrt{2}/2\right)$; slope: $m = \sqrt{2}/2$

eq. of line: $y - \sqrt{2}/2 = \sqrt{2}/2\left(x - \frac{\pi}{4}\right)$

answer: $\boxed{L(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right)}$

(b) Use your linearization to approximate $\sin\left(\frac{\pi}{4} + \frac{1}{10}\right)$. Express your answer as a single fraction.

$$L\left(\frac{\pi}{4} + \frac{1}{10}\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(\frac{\pi}{4} + \frac{1}{10} - \frac{\pi}{4}\right)$$

$$= \frac{\sqrt{2}}{2}\left(1 + \frac{1}{10}\right) = \frac{\sqrt{2}}{2}\left(\frac{11}{10}\right) = \boxed{\frac{11\sqrt{2}}{20}}$$

6. (15 points) Air is being slowly released from a spherical balloon. At time t = 0, the radius of the balloon is observed to be 10 cm and the radius is observed to be decreasing at the rate of 1 cm/s.

(a) Determine the rate of change of volume of the balloon when $r = 10$.

at $t=0$, $r=10$ cm

$\frac{dr}{dt} = -1 \ \frac{cm}{s}$

goal: Find $\frac{dV}{dt}$ when $r=10$.

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

plug in

$\frac{dV}{dt} = 4\pi(10)^2(-1)$

$= \boxed{-400\pi \ cm^3/s}$

(b) Assuming the rate of change of volume remains constant, how long will it take to empty the balloon?

at $t=0$, $r=10$; So $V = \frac{4}{3}\pi 10^3 = \frac{4000\pi}{3} \ cm^3$ when $t=0$.

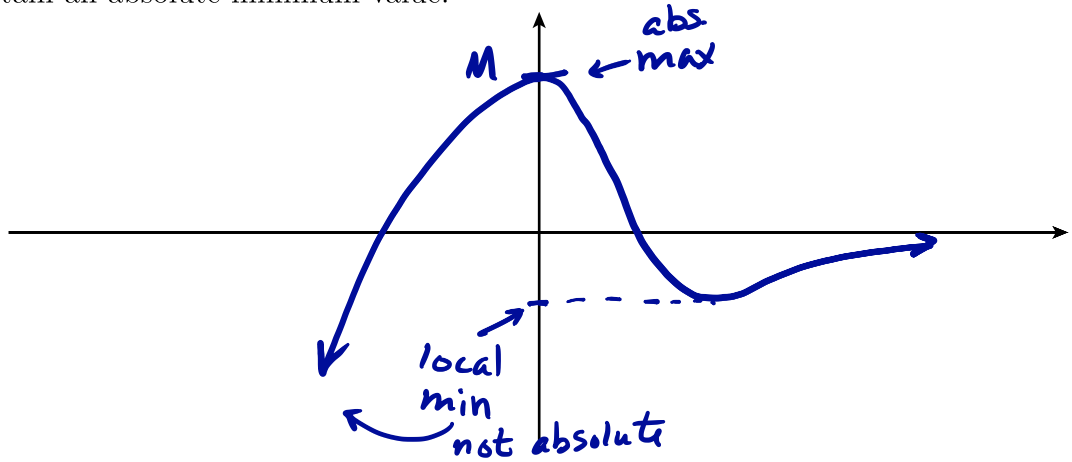
$\frac{dV}{dt} = -400\pi \ cm^3/s$ (i.e. Each second, the volume V is decreasing by $400\pi \ cm^3$)

How long to empty?

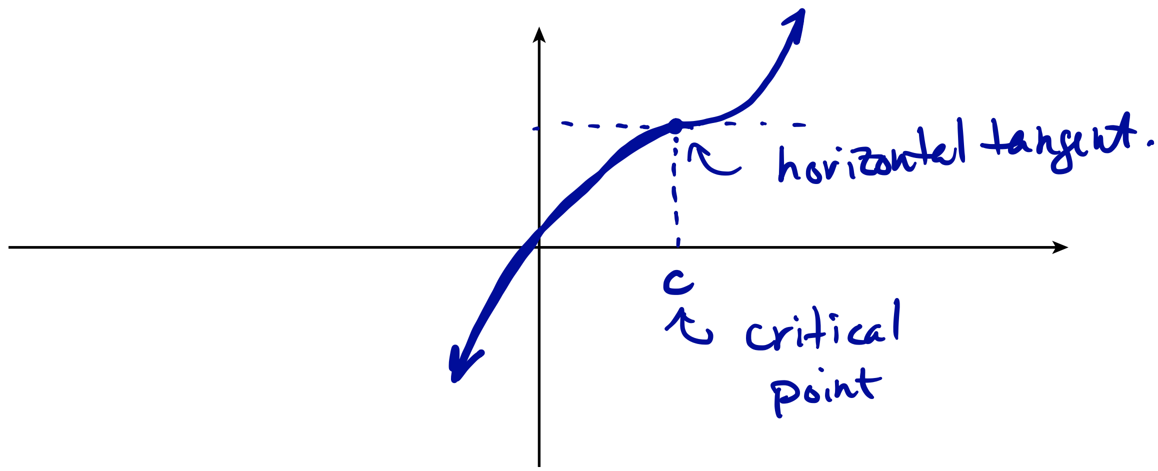
$\frac{V}{\left|\frac{dV}{dt}\right|} = \frac{4000\pi/3}{400\pi} = \boxed{\frac{10}{3} \text{ sec.}}$

7. (10 points) For each of the following scenarios, draw the graph of a function $f(x)$ with domain all of $\mathbb{R}$ that has a derivative at every point and that satisfies the desired criteria.

(a) The function attains an absolute maximum value and has a local minimum, but does not attain an absolute minimum value.



(b) The function has a critical point, but at no point has a local minimum or maximum value.



[**5 points extra credit:**] Formally state the Mean Value Theorem and use it to prove that for all real numbers a and b where $a < b$,

$$-(b-a) \leq \sin b - \sin a \leq b - a.$$

If $f(x)$ is conts on $[a,b]$ and differentiable on (a,b), then there is a c in (a,b) so that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Pick $f(x) = \sin x$. So $f'(x) = \cos x$. Also, f is continuous + differentiable for all $\mathbb{R}$. Now MVThm says there is c in (a,b) so that:

$$\cos c = \frac{\sin(b) - \sin(a)}{b-a} \quad \text{or} \quad (b-a)\cos c = \sin b - \sin a.$$

But $-1 \leq \cos c \leq 1$.

So: $-(b-a) \leq \sin b - \sin a \leq (b-a)$.