Math F251 Midterm 1 Spring 2019

Name: SOLUTIONS

Section: ☐ F01 (Rhodes) ☐ F02 (Bueler)

Rules:

You have 60 minutes to complete the exam.

Partial credit will be awarded, but you must show your work.

No calculators, books, notes, or other aids are permitted.

Circle your final answer to each question where appropriate.

If you need extra space, you can use the back sides of the pages. (Clearly label any work you want graded.)

Turn off anything that might go beep during the exam.

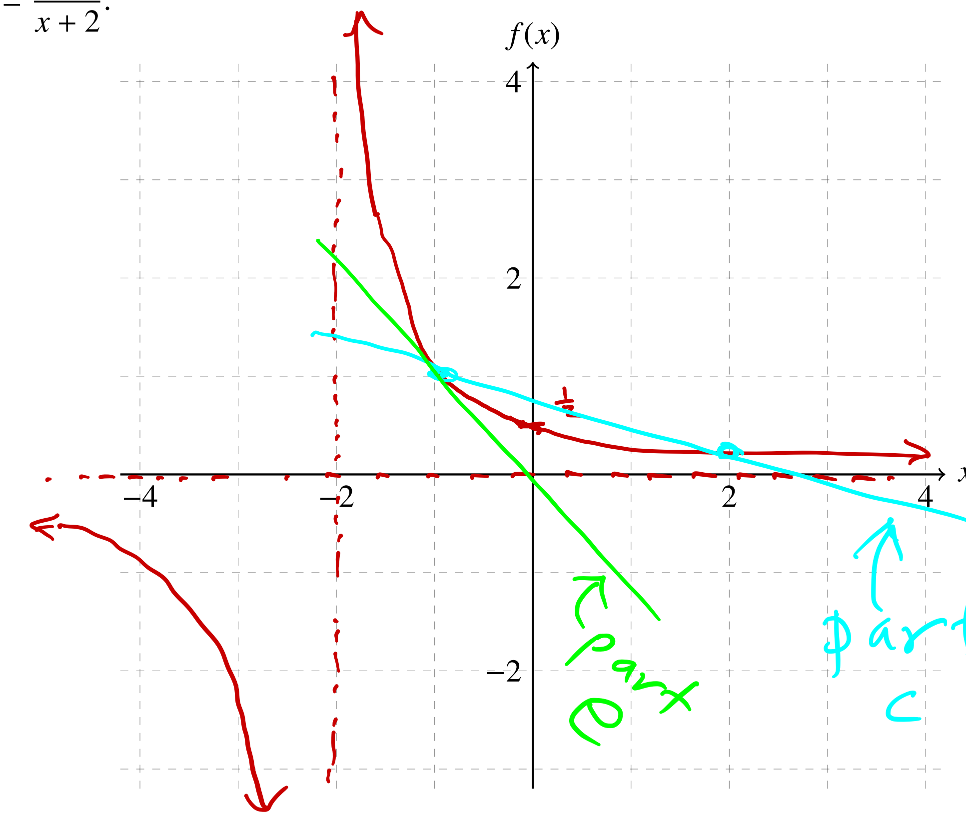
Good luck!

Problem	Possible	Score
1	20	
2	12	
3	20	
4	12	
5	15	
6	16	
Extra Credit	3	
Total	95	

1. (20 points)

Consider the function $f(x) = \dfrac{1}{x+2}$.

a. Sketch its graph.



b. The graph in **a.** contains a vertical asymptote. Clearly state a limit that justifies this vertical asymptote.

$\lim_{x \to -2^+} f(x) = +\infty$ so $x = -2$ is a vertical asymptote

c. Find the average slope over the interval $[-1, 2]$.

$$m_{av} = \frac{f(2) - f(-1)}{2 - (-1)} = \frac{\frac{1}{2+2} - \frac{1}{-1+2}}{3}$$

$$= \frac{\frac{1}{4} - 1}{3} = \frac{-3/4}{3} = -1/4$$

Continued....

d. Recall that the function is $f(x) = \frac{1}{x+2}$. Using the definition of the derivative, find $f'(-1)$. Make sure to use proper limit notation. [**No credit** will be awarded for computing the derivative using any approach other than the limit definition.]

$$f'(-1) = \lim_{h\to 0} \frac{f(-1+h)-f(-1)}{h} = \lim_{h\to 0} \frac{\frac{1}{-1+h+2} - \frac{1}{-1+2}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{1+h} - 1}{h} = \lim_{h\to 0} \frac{1-(1+h)}{(1+h)h}$$

$$= \lim_{h\to 0} \frac{-\cancel{h}}{(1+h)\cancel{h}} = \lim_{h\to 0} \frac{-1}{1+h} = \frac{-1}{1+0} = -1$$

e. Find an equation of the tangent line at $x = -1$.

$$f(-1) = \frac{1}{-1+2} = 1 \quad \therefore \quad y - 1 = (-1)(x-(-1))$$

[or $y = -x$]

2. (12 points)

Suppose

$$f(x) = \frac{x^2 + x - 12}{x - 3}$$

a. What is the domain of f? Use interval notation.

domain $= \{x \neq 3\} = (-\infty, 3) \cup (3, \infty)$

b. How would you extend the definition of the function f in order to make it continuous at $x = 3$?

$$f(x) = \frac{(x-3)(x+4)}{x-3} \underset{x \neq 3}{=} x+4$$

so extend this way $f(3) = 7$

c. Using a limit, write a mathematical statement expressing that the extended function is continuous at $x = 3$.

$$\lim_{x\to 3} f(x) = f(3)$$

(and both sides equal 7)

3. (20 points)

Compute the following limits, or explain why the limit does not exist. (Show appropriate work and use proper limit notation for full credit.)

a. $\lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}$ $= \lim_{h\to 0} \frac{\cancel{x^3}+3x^2h+3xh^2+h^3-\cancel{x^3}}{h}$

$= \lim_{h\to 0} 3x^2+3xh+h^2 = 3x^2+0+0 = 3x^2$

b. $\lim_{x\to\infty} \frac{1-2x^2}{x^2+x+4}$ $= \lim_{x\to\infty} \frac{1-2x^2}{x^2+x+4} \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$

$= \lim_{x\to\infty} \frac{\frac{1}{x^2}-2}{1+\frac{1}{x}+\frac{4}{x^2}} = \frac{0-2}{1+0+0} = -2$

c. $\lim_{y\to\pi^-} \frac{\cos(y)}{y+1}$ $= \frac{\cos(\pi)}{\pi+1} = \frac{-1}{\pi+1}$

(function is continuous except at $y=-1$)

d. $\lim_{x\to 4} \frac{4-x}{\sqrt{x^2+9}-5}$ $= \lim_{x\to 4} \frac{4-x}{\sqrt{x^2+9}-5} \frac{\sqrt{x^2+9}+5}{\sqrt{x^2+9}+5}$

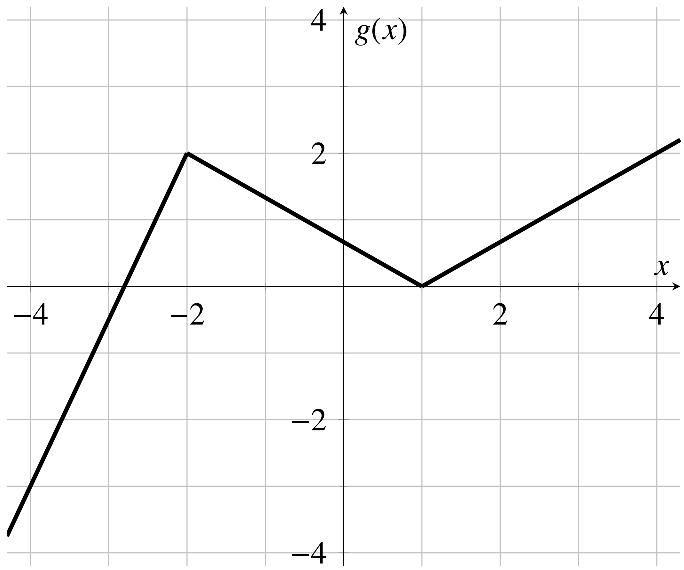
$= \lim_{x\to 4} \frac{(4-x)(\sqrt{x^2+9}+5)}{(x^2+9)-25} = \lim_{x\to 4} \frac{(4-x)(\sqrt{x^2+9}+5)}{x^2-16}$

$= \lim_{x\to 4} \frac{-\cancel{(x-4)}(\sqrt{x^2+9}+5)}{\cancel{(x-4)}(x+4)} = \frac{-(\sqrt{16+9}+5)}{4+4} = -\frac{10}{8}$

$= -\frac{5}{4}$

4. (12 points)

Suppose the graph of $g(x)$ is as shown.



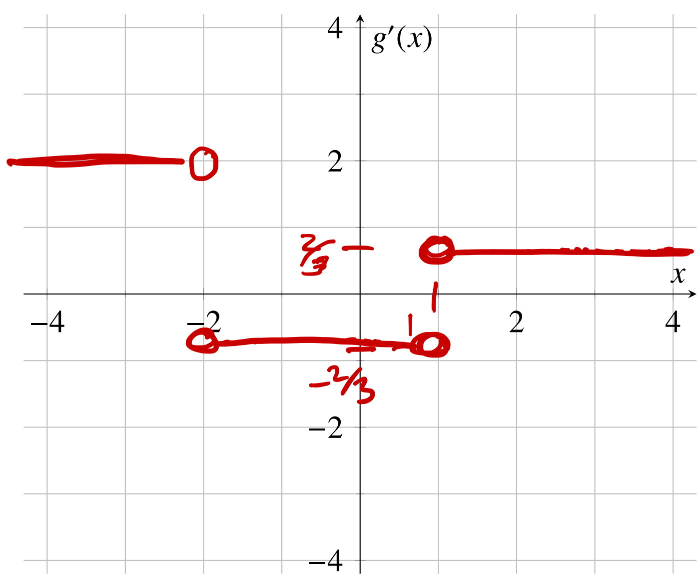
a. From the graph of g above, evaluate the derivatives:

$g'(0) =$ $-\frac{2}{3}$, $g'(3) =$ $+\frac{2}{3}$

b. Identify the two values where the derivative does not exist:

$x =$ -2 , 1

c. On the axes below, sketch the graph of $g'(x)$. (It must be consistent with parts **a.** and **b.**)



5. (15 points)

A ball is dropped from a cliff at time $t = 0$, and its height in feet at t seconds is

$$s(t) = 1000 - 16t^2.$$

When answering the following questions, be sure to include **units** *where appropriate*.

a. What is the height of the ball at time $t = 2$?

$$S(2) = 1000 - 16 \cdot 2^2 = 1000 - 64 = 936 \text{ feet}$$

b. Compute the average velocity (i.e. the average rate of change of the height) from $t = 1$ to $t = 2$.

$$m_{av} = \frac{S(2) - S(1)}{2-1} = \frac{(\cancel{1000} - 16 \cdot 2^2) - (\cancel{1000} - 16 \cdot 1^2)}{1} = -64 + 16 = -48 \frac{ft}{s}$$

c. Using the definition of the derivative, compute $s'(2)$. Make sure to use proper limit notation. [**No credit** will be awarded for computing the derivative using any approach other than the limit definition.]

$$s'(2) = \lim_{h \to 0} \frac{s(2+h) - s(2)}{h}$$

$$= \lim_{h \to 0} \frac{(\cancel{1000} - 16(2+h)^2) - (\cancel{1000} - 16(2)^2)}{h}$$

$$= \lim_{h \to 0} \frac{-16(\cancel{4} + 4h + h^2) + \cancel{16 \cdot 4}}{h} = \lim_{h \to 0} \frac{-16(4h + h^2)}{h}$$

$$= \lim_{h \to 0} -16(4+h) = -16(4+0) = -64 \, ft/s$$

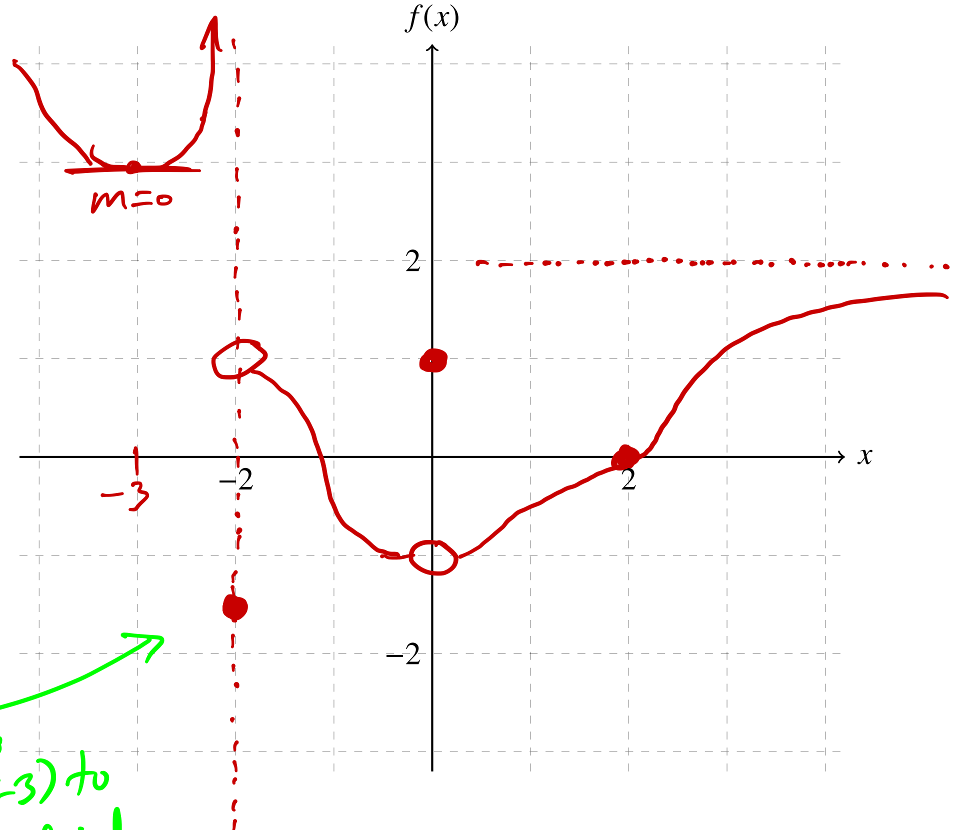
d. What is the instantaneous velocity of the ball at time $t = 2$?

$-64 \, ft/s$

6. (16 points)

On the axes at right, sketch the graph of a function $f(x)$ satisfying the following criteria:

1. $f(0) = 1$
2. $f(2) = 0$
3. $\lim_{x\to 0} f(x) = -1$
4. $\lim_{x\to -2^-} f(x) = +\infty$
5. $\lim_{x\to -2^+} f(x) = 1$
6. $\lim_{x\to +\infty} f(x) = 2$
7. $f'(-3) = 0$
8. domain of f is $(-\infty, \infty)$

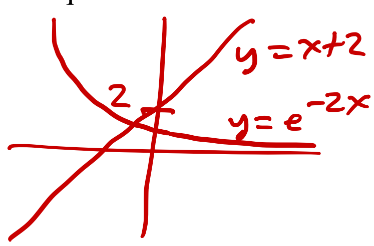


requires $f(-3)$ to be defined

Extra Credit. (3 points)

Use the Intermediate Value Theorem to show that the curves $y = e^{-2x}$ and $y = x+2$ cross, that is, that there is an x-value where they are equal.

informally:



solution: Let $f(x) = e^{-2x} - (x+2)$. Then f is continuous everywhere. And

$$f(0) = e^{0} - (0+2) = -2 < 0$$

$$f(-1) = e^{2} - (-1+2) = e^{2} - 1 > 0 \quad [e^2 > 2^2 = 4]$$

By I.V.T. there is c in $(-1, 0)$ so that $f(c) = 0$. Then $e^{-2c} = c+2$.