Math F251 Midterm 2 Spring 2019

Name: SOLUTIONS

Section: ☐ F01 (Rhodes)
☐ F02 (Bueler)

Rules:

You have 60 minutes to complete the exam.

Partial credit will be awarded, but you must show your work.

No calculators, books, notes, or other aids are permitted.

Circle your final answer to each question where appropriate.

If you need extra space, you can use the back sides of the pages. (Clearly label any work you want graded.)

Turn off anything that might go beep during the exam.

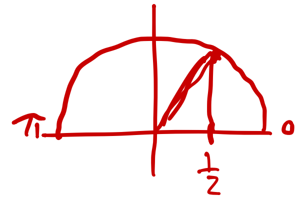
Good luck!

Problem	Possible	Score
1	10	
2	10	
3	10	
4	10	
5	15	
6	15	
7	10	
8	10	
Extra Credit	3	
Total	90	

1. (10 points)

Consider the function

$$f(x) = x - 2\sin(x).$$



a. Find the critical number(s) of $f(x)$ on the interval $[0, \pi]$

$$f'(x) = 1 - 2\cos(x) = 0$$

$$\cos(x) = \frac{1}{2} \therefore x = \frac{\pi}{3}$$

b. Find the absolute maximum and absolute minimum values of $f(x)$ on the interval $[0, \pi]$.

	x	$f(x)$
	0	0
abs min here $\rightarrow$	$\pi/3$	$\frac{\pi}{3} - 2\sin\left(\frac{\pi}{3}\right) = \frac{\pi}{3} - \sqrt{3} < 0$
abs max here $\rightarrow$	π	π

$$\text{abs. min} = f(\pi/3) = \frac{\pi}{3} - \sqrt{3}$$

$$\text{abs. max} = f(\pi) = \pi$$

2. (10 points)

a. Find the linearization of $g(x) = \sqrt{x}$ at $a = 4$.

$$g'(x) = \frac{1}{2\sqrt{x}}$$

$$L(x) = g(a) + g'(a)(x - a)$$

$$= \sqrt{4} + \frac{1}{2\sqrt{4}}(x - 4) = 2 + \frac{1}{4}(x - 4)$$

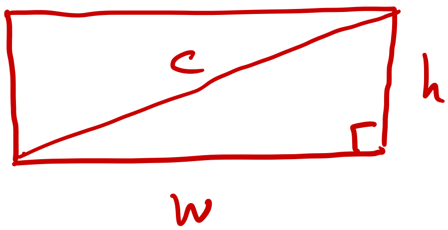
b. Use your result in part **a.** to approximate $\sqrt{3.9}$.

$$g(3.9) \approx L(3.9) = 2 + \frac{1}{4}(3.9 - 4)$$

$$= 2 - \frac{1}{40} = \frac{79}{40}$$

3. (10 points)

A rectangular photograph whose width is 3 times the height is being enlarged, and the height is increasing at a rate of 2 in/min. How fast is the diagonal of the rectangle increasing when the height is 4 in? Indicate units.



$w = 3h$

$c^2 = w^2 + h^2 = (3h)^2 + h^2$

$c^2 = 10h^2$

$c = \sqrt{10}\, h$

$$\frac{dc}{dt} = \sqrt{10}\,\frac{dh}{dt} = \sqrt{10}\cdot 2 \ \frac{\text{in}}{\text{min}}$$

4. (10 pts.)

Evaluate the following limits.

a. $\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}$ $\overset{\frac{\infty}{\infty}}{\underset{L'H.}{=}} \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} \frac{2\sqrt{x}}{x}$

$= \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0$

b. $\lim_{\theta\to\pi/2} \frac{1-\sin\theta}{1+\cos(2\theta)}$ $\overset{\frac{0}{0}}{\underset{L'H}{=}} \lim_{\theta\to\frac{\pi}{2}} \frac{-\cos\theta}{-2\sin(2\theta)} = \lim_{\theta\to\frac{\pi}{2}} \frac{\cos\theta}{2\sin(2\theta)}$

$\overset{\frac{0}{0}}{\underset{L'H}{=}} \lim_{\theta\to\frac{\pi}{2}} \frac{-\sin\theta}{4\cos(2\theta)} = \frac{-1}{4\cdot(-1)} = \frac{1}{4}$

5. (15 points)

Consider the curve

$$y = \frac{x^2}{x^2 + 5}.$$

a. Determine the domain, and symmetry (if any).

domain: $(-\infty, \infty)$

symmetry: even

b. Find all asymptotes (horizontal and vertical). For each asymptote compute a limit which justifies that asymptote.

no vertical asymptotes

$$\lim_{x \to \infty} \frac{x^2}{x^2+5} = 1 \quad \text{so} \quad y = 1$$

is horizontal asymptote

can be justified by L'Hopital's rule

c. Find all intervals of increase and decrease.

$$f'(x) = \frac{2x(x^2+5) - x^2(2x)}{(x^2+5)^2} = \frac{\cancel{2x^3} + 10x - \cancel{2x^3}}{(x^2+5)^2}$$

$$= \frac{10x}{(x^2+5)^2} \qquad f'(x) \geq 0 \iff x \geq 0$$

increase: $[0, \infty)$

decrease: $(-\infty, 0]$

Problem 5 continued....

d. Find all points of inflection.

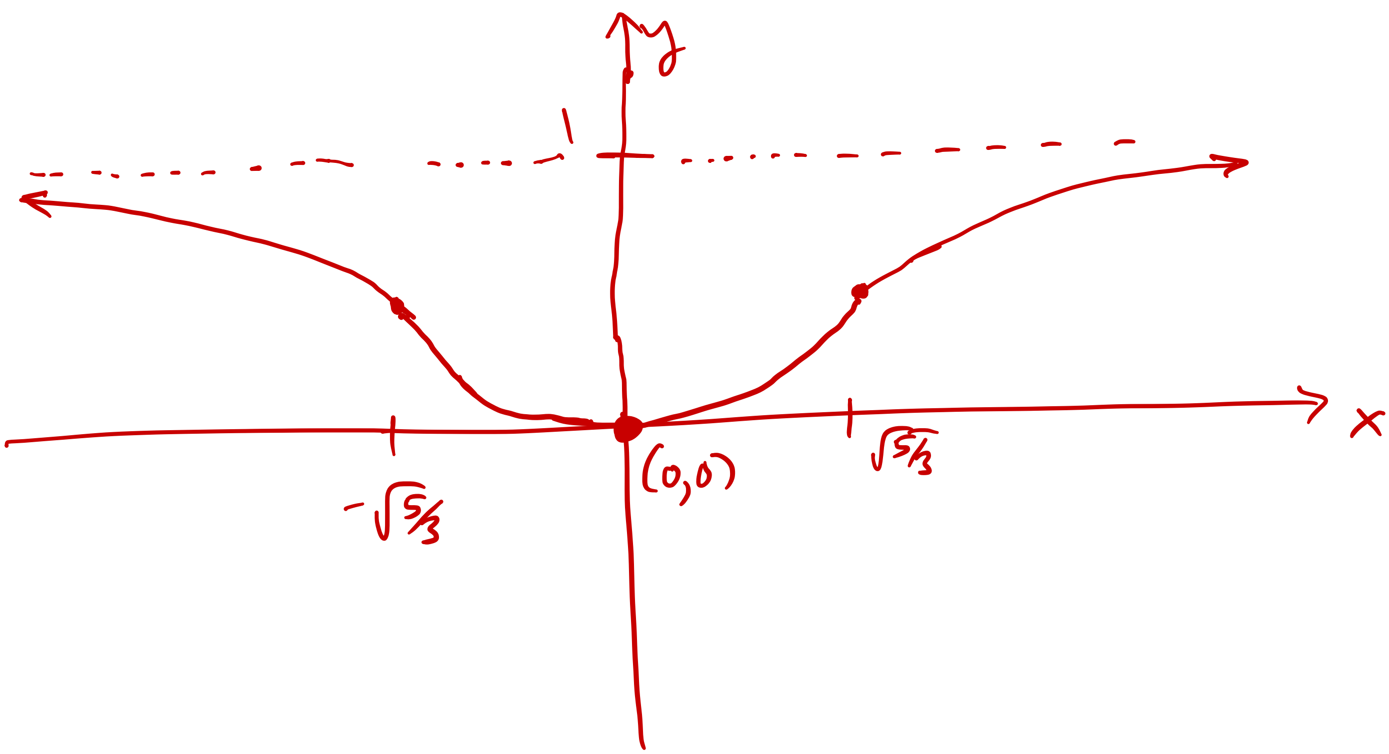
$$f''(x) = \frac{10(x^2+5)^2 - 10x\cdot 2(x^2+5)\cdot 2x}{(x^2+5)^4} = \frac{10(x^2+5)\left[x^2+5-4x^2\right]}{(x^2+5)^4}$$

$$= \frac{10(5-3x^2)}{(x^2+5)^3}$$

$$f''(x)=0 \iff 5-3x^2=0$$

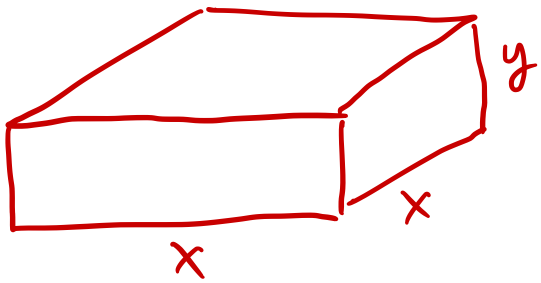
$$x = \pm\sqrt{\frac{5}{3}}$$ pts of infl.

e. Sketch the graph, labeling all important points.



6. (15 points)

An open-topped box with a square base is to be made so that its volume is 4000 cm^3. Find the dimensions of the box using the least materials. Indicate units.

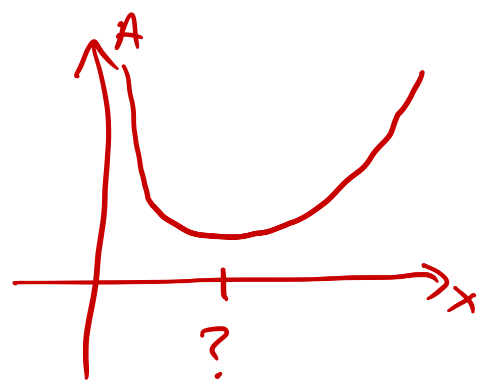


$$4000 = x^2 y$$

$$A = \underbrace{x^2}_{\text{base}} + \underbrace{4xy}_{\text{sides}}$$

$$y = \frac{4000}{x^2} \quad \therefore \quad A(x) = x^2 + \frac{16000}{x}$$

$$A'(x) = 2x - \frac{16000}{x^2}$$



$$A'(x) = 0 \iff 2x^3 = 16000$$

$$\iff x^2 = 8000$$

$$\iff x = 20 \implies y = \frac{4000}{x^2} = \frac{4000}{400} = 10$$

base is 20 cm × 20 cm

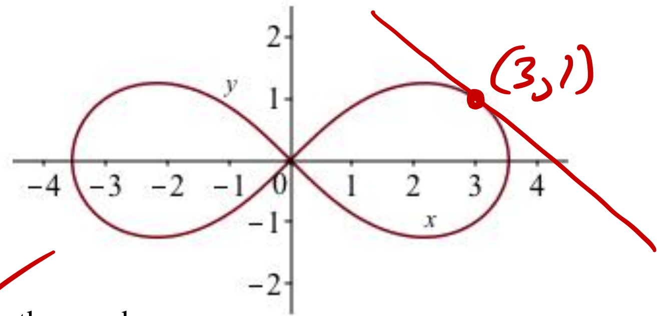
height is 10 cm

7. (10 pts.)

A racetrack is described by the equation

$$2(x^2 + y^2)^2 = 25(x^2 - y^2),$$

whose graph is shown. If an object falls off a speeding car as it travels over the track, it will follow a tangent line.



a. Verify that (3, 1) is a point on the track, and mark it on the graph.

$$2(3^2+1)^2 \overset{?}{=} 25(3^2-1)$$

$$2(10)^2 \overset{?}{=} 25(8)$$

$$200 = 200$$

b. Draw the tangent line at (3,1) on the graph.

c. Find an equation of the tangent line at (3,1). implicit differentiati

$$4(x^2+y^2)\left(2x+2y\frac{dy}{dx}\right) = 25\left(2x-2y\frac{dy}{dx}\right)$$

$$8x(x^2+y^2) + 8y(x^2+y^2)\frac{dy}{dx} = 50x - 50y\frac{dy}{dx}$$

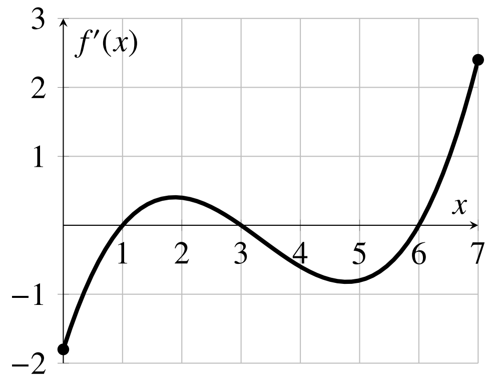
$$\frac{dy}{dx} = \frac{50x - 8x(x^2+y^2)}{8y(x^2+y^2)+50y}$$

$$m = \left.\frac{dy}{dx}\right|_{(3,1)} = \frac{150 - 24(10)}{8(10)+50} = \frac{-90}{130} = \frac{-9}{13}$$

$$y - 1 = \frac{-9}{13}(x-3)$$

8. (10 points)

The graph of the **derivative** f' of a function f is shown.



a. On what intervals is f increasing or decreasing? Use interval notation.

increasing: $[1,3] \cup [6,7]$

decreasing: $[0,1] \cup [3,6]$

b. At what values of x does f have a local maximum or minimum?

loc. max. $x=3$

loc. min $x=1$, $x=6$

Extra Credit. (3 points)

For $x > 0$, let $g(x) = x^x$. Compute the first and second derivatives of g.

$$g(x) = x^x$$

$$\ln g(x) = x \ln x$$

$$\frac{1}{g(x)} g'(x) = 1 \cdot \ln x + x \cdot \frac{1}{x} = 1 + \ln x$$

$$g'(x) = (1+\ln x) g(x) = (1+\ln x) x^x$$

$$\ln g'(x) = \ln(1+\ln x) + x \ln x$$

$$\frac{1}{g'(x)} g''(x) = \frac{1}{1+\ln x} + 1 \cdot \ln x + x \cdot \frac{1}{x}$$

$$g''(x) = \left(\frac{1}{1+\ln x} + 1 + \ln x\right)(1+\ln x) x^x$$

$$= \left(1 + (1+\ln x)^2\right) x^x$$