Math F251 Final Exam Spring 2020

Name: Solutions

Section: ☐ F01 (Faudree)
☐ F02 (Bueler)
☐ UX1 (Van Spronsen)

All students must affirm the following statements by initialing in the blanks provided. Students using their own paper must write out the statements in full.

__________ I will not seek or accept help from anyone.

__________ I will not use a calculator, books, notes, the internet or other aids.

__________ I understand that answers without work will not be awarded credit.

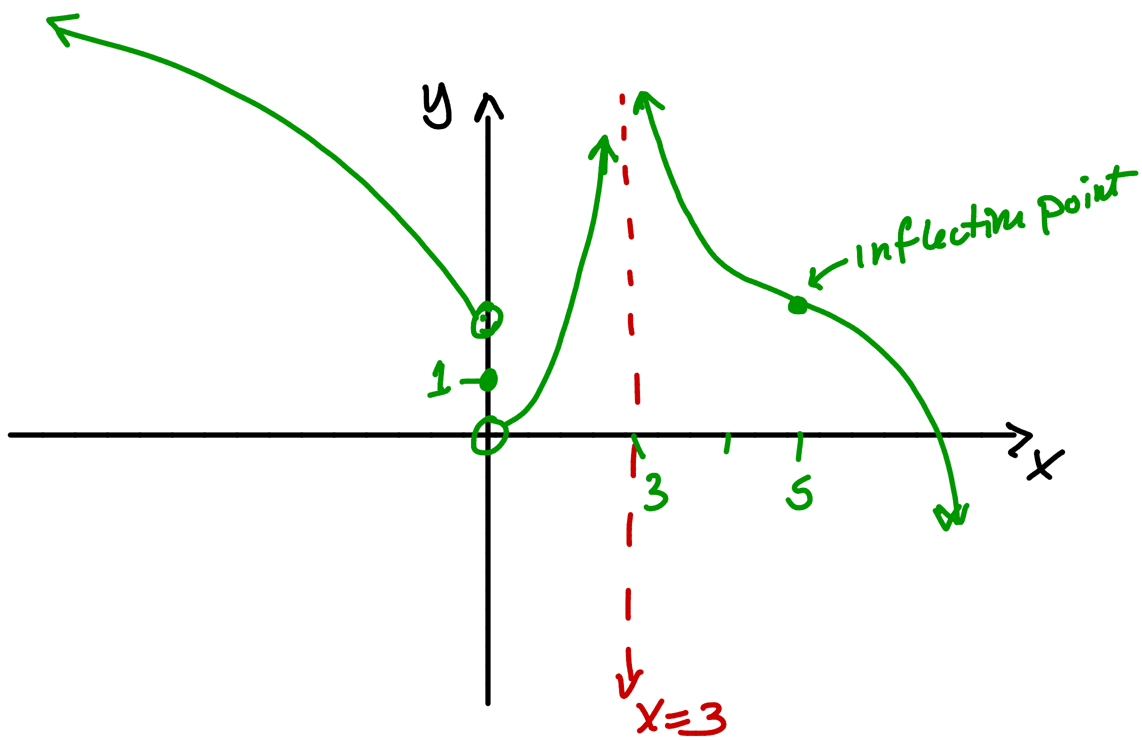
Good luck!

Problem	Possible	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total	100	

1. (10 points)

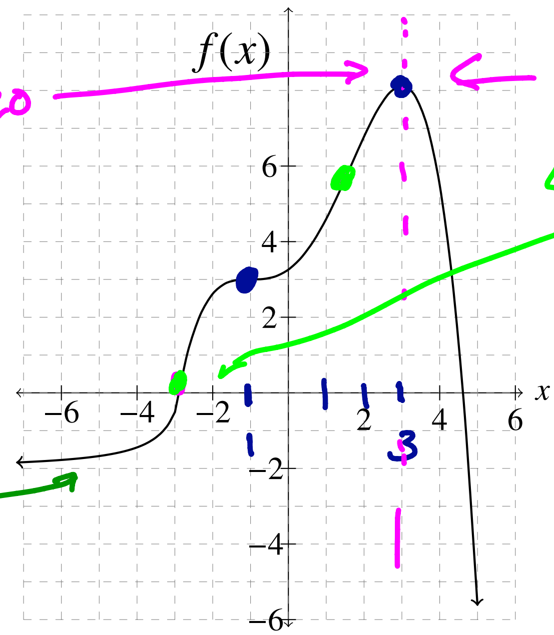
Sketch a graph $H(x)$ with all of the properties below. Label your graph.

- The domain of $H(x)$ is $(-\infty, 3) \cup (3, \infty)$.
- $H(0) = 1$ point (0,1)
- $\lim_{x\to 0^-} H(x) = 2$ graph $\to$ y=2 on left of x=0
- $\lim_{x\to 0^+} H(x) = 0$ graph $\to$ y=0 on right of x=0
- $\lim_{x\to 3} H(x) = \infty$ x=3 is a vertical asymptote
- $H'(x) < 0$ and $H''(x) < 0$ on the interval $(-\infty, 0)$ left of x=0, graph is $\downarrow$ and $\frown$
- H has an inflection point when $x = 5$



2. (10 points)

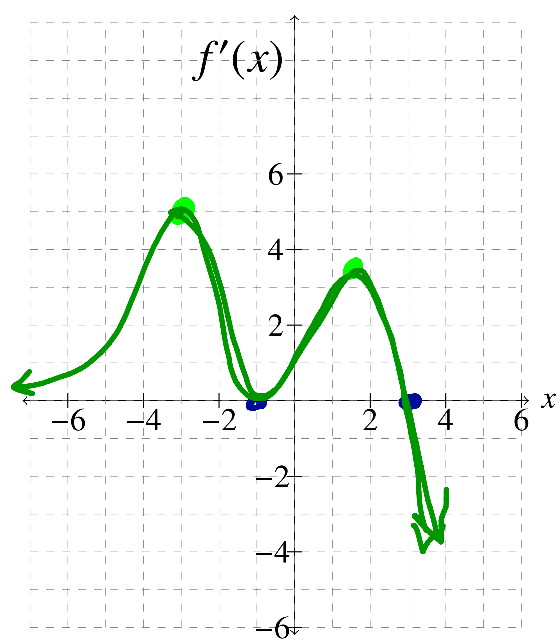
The graph of $f(x)$ is sketched below.



1. List the x-values of all critical numbers of f.

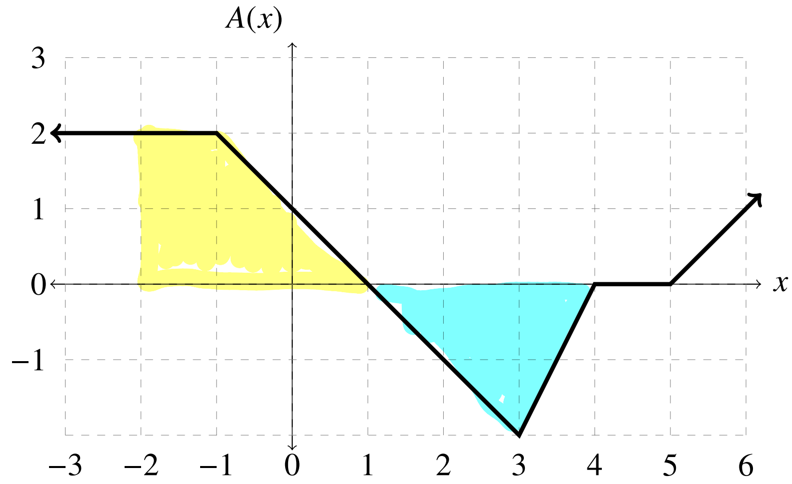
   $x = -1$ and $x = 3$ (see blue dots)

2. Use the graph of $f(x)$ to sketch the graph of $f'(x)$ on the set of axes below.



3. (10 points)

The function $A(x)$ is graphed below.



a. $A(0) =$ 1

b. $A'(0) =$ −1 (slope)

c. At what x values, if any, does $A'(x)$ not exist?

at corners: $x = -1, 3, 4, 5$

d. By using your knowledge of areas, evaluate $\int_{-2}^{4} A(x)\,dx$. $= \left(\text{yellow area}\right) - \left(\text{blue area}\right)$

$= 4 - 3 = 1$

For parts (e)-(g), let $H(x) = \displaystyle\int_0^x A(s)\,ds$.

e. What is the value of $H(2)$?

$H(2) = \int_0^2 A(s)ds = \frac{1}{2} - \frac{1}{2} = 0$

f. What is the value of $H'(2)$?

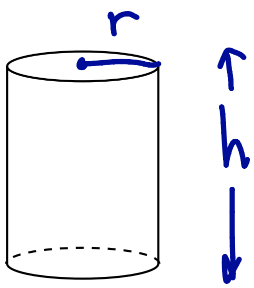
$H'(2) = A(2) = -1$

g. Where on the interval $[0, 6]$ is $H(x)$ decreasing?

Where $A(s) < 0$ or on the interval $(1, 4)$

4. (10 points)

The height of a right circular cylinder is increasing at rate of 3 meters per second while its volume remains constant. (See figure below.) At what rate is the radius changing when the radius and height are both 10 meters?



$\frac{dr}{dt} = 3, \quad \frac{dV}{dt} = 0$

Find $\frac{dr}{dt}$ when $r = h = 10$.

$V = \pi r^2 h$

$\frac{dV}{dt} = \pi \left(2r \frac{dr}{dt} h + r^2 \frac{dh}{dt} \right)$

$0 = \pi \left(2 \cdot 10 \cdot \frac{dr}{dt} \cdot 10 + 10^2 \cdot 3 \right) = 300 + 200 \frac{dr}{dt}$

$\frac{dr}{dt} = -\frac{3}{2}$ m/s.

The radius is decreasing at a rate of $3/2$ m/s.

5. (10 points)

Find any horizontal or vertical asymptotes for the function $f(x) = \frac{2x^2-3x}{5x^2-10}$. Use limits to justify your answer(s). If no asymptote exists, explain why.

$$\boxed{\begin{array}{l}\text{Answer: VA: } x=\sqrt{2},\ x=-\sqrt{2} \\ \qquad\quad\ \text{HA: } y = 2/5\end{array}}$$

Justification:

$$\lim_{x\to\infty} \frac{2x^2-3x}{5x^2-10} = \lim_{x\to\infty} \frac{2-3/x}{5-10/x^2} = \frac{2}{5}$$

$$\lim_{x\to-\infty} \frac{2x^2-3x}{5x^2-10} = \frac{2}{5} \text{ (by same algebra)}$$

$$\lim_{x\to\sqrt{2}^+} \frac{2x^2-3x}{5x^2-10} = -\infty$$

as $x\to\sqrt{2}^+$, $5x^2-10\to 0^+$ and $2x^2-3x\to 4-3\sqrt{2}<0$

$$\lim_{x\to-\sqrt{2}^-} \frac{2x^2-3x}{5x^2-10} = +\infty$$

as $x\to-\sqrt{2}^-$, $5x^2-10\to 0^+$ and $2x^2-3x\to 4+3\sqrt{2}>0$

6. (10 points)

A homeowner wants to minimize the cost of heating a building over the next 10 years. Adding x inches of insulation in the attic costs \$100 per inch and results in heating costs of $1000/(2+x)$ dollars over 1 year. How many inches of insulation should be installed in order the minimize the total costs over a 10 year period? Justify your answer. (By **total costs**, we mean both the initial cost of insulating the building plus the annual heating costs.)

goal: minimize cost.

$$C(x) = \underset{\uparrow\ \text{cost of installation}}{100x} + \underbrace{10\left(\frac{1000}{2+x}\right)}_{\text{cost of 10 years of heating}} = 100x + 10{,}000(2+x)^{-1}, \quad \text{domain: } (0,\infty)$$

$$C'(x) = 100 - 10{,}000(2+x)^{-2} = 0$$

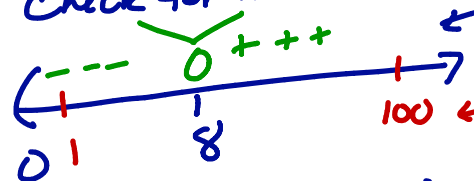
$$100 = \frac{10{,}000}{(2+x)^2}$$

$$(2+x)^2 = 100$$

$$2+x = +10 \quad \text{b/c domain}$$

$$x = 8$$

Check for minimum:



sign of C'

sample points

So, C has a min at $x=8$. The min is absolute because it is the unique c# in domain.

ANS: 8 inches of insulation.

7. (10 points)

Evaluate the integrals below. Note that these problems will be graded **largely** by the quality of the work written. So make sure to include proper notation and compete steps.

a. $\int \sin(2x) + \frac{(1 + \ln x)^2}{x} dx$ $= \int \sin(2x)dx + \int (1+\ln x)^2 (x^{-1}dx)$

$= -\frac{1}{2}\cos(2x) + \frac{1}{3}(1+\ln x)^3 + C$

$\boxed{\begin{array}{l} u=2x \\ du=2dx \\ \frac{1}{2}du=dx \end{array}}$

$\int \sin(2x)\,dx = \int \sin(u)\cdot\frac{1}{2}du$

$= -\frac{1}{2}\cos(u)+C$

$= -\frac{1}{2}\cos(2x)+C$

$\boxed{\begin{array}{l} u=1+\ln x \\ du=\frac{1}{x}dx \end{array}}$

$\int (1+\ln x)^2 (x^{-1}dx) = \int u^2 du$

$= \frac{1}{3}u^3 + C = \frac{1}{3}(1+\ln x)^3 + C$

b. $\int_0^2 (1 + xe^{\pi x^2})dx$ $= x\Big]_0^2 + \int_0^2 xe^{\pi x^2} dx$

$\boxed{\begin{array}{l} \text{let} \\ u=\pi x^2 \\ du=2\pi x\,dx \\ \frac{1}{2\pi}du = x\,dx \end{array}}$

if $x=0$, $u=0$

$x=2$, $u=4\pi$

$= 2-0 + \int_0^{4\pi} e^u \cdot \frac{1}{2\pi} du$

$= 2 + \frac{1}{2\pi} e^u\Big]_0^{4\pi}$

$= 2 + \frac{1}{2\pi}\left(e^{4\pi} - e^0\right) = 2 + \frac{1}{2\pi}\left(e^{4\pi} - 1\right)$

8. (10 points)

Use f, f' and f'' to answer the questions below.

$$f(x) = x\sqrt[3]{x^2-5} \qquad f'(x) = \frac{5(x^2-3)}{3(x^2-5)^{2/3}} \qquad f''(x) = \frac{10x(x^2-9)}{9(x^2-5)^{5/3}}.$$

a. Determine all critical numbers of the function f. Show how you obtain your answer.

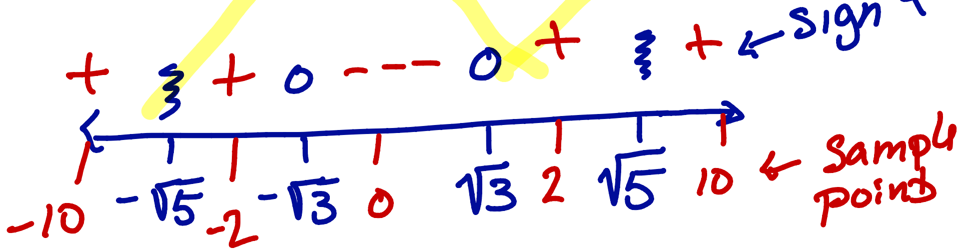
$f'=0$ when $x=\pm\sqrt{3}$

f' undefined when $x=\pm\sqrt{5}$

answer: $x=-\sqrt{5}, -\sqrt{3}, \sqrt{3}, \sqrt{5}$

b. For each critical number of f, classify it as a local minimum, a local maximum or neither. Show how you obtain your answer.

use 1st der test:



* sign of f' determined by (x^2-3)-term alone. *

answer:

local max at $x=-\sqrt{3}$

local min at $x=\sqrt{3}$

neither at

$x=\sqrt{5}$

$x=-\sqrt{5}$

9. (10 points)

Short Answer

a. A population of chickadees is increasing at a rate of $r(t)$ chickadees per year. What does $\int_1^4 r(t)\,dt = 400$ mean? Make sure to include units in your answer.

Between years 1 and 4, the population of chickadees increased by 400 birds.

b. Let $y = -3 + 5(x - 4)$ be an equation of the tangent line to the graph of $f(x)$ at $x = 4$. Is it possible to determine $f(4)$ or $f'(4)$? Explain your answer.

$f(4) = -3 + 5(4-4) = -3$ since $f(x)$ and the tangent line share the point of tangency.

$f'(4) = 5$ since $f(x)$ and the tangent line have the same slope at the point of tangency

c. Let $C(T)$ be the number of chirps per second of a male cricket as a function of temperature, T, in degrees Fahrenheit. In the context of the problem, interpret $C'(70) = 2$. Make sure to include units in your answer.

units of C' : chirps/sec/°F

At 70°F, the rate of change of chirps per second is increasing at a rate of 2 chirps per second per °F.

(or) If temperature increases by 1 degree Fahrenheit, from 70°F to 71°F, we estimate the crickets will increase the number of chirps per second by 2.

10. (10 points)

The acceleration function (in m/s^2), the initial velocity, and the initial position are given for a particle moving along a line. Find an expression for position, s, at time t.

$$a(t) = \frac{12}{(1+t)^3}, \quad v(0) = 0, \quad s(0) = 0$$

$a(t) = 12(1+t)^{-3}$

$v(t) = \int 12(1+t)^{-3}\,dt = 12(1+t)^{-2}(\frac{1}{-2}) + C = -6(1+t)^{-2} + C$

$0 = v(0) = -6(-1)^{-2} + C$. So $0 = -6 + C$. So $C = 6$.

$v(t) = -6(1+t)^{-2} + 6$

$s(t) = \int v(t)\,dt = \int (-6(1+t)^{-2} + 6)\,dt = 6(1+t)^{-1} + 6t + C$

$0 = s(0) = 6(1+0)^{-1} + 0 + C$. So $0 = 6 + C$. So $C = -6$.

$$\boxed{\underline{\text{ANS}}: \; s(t) = \frac{6}{1+t} + 6t - 6}$$