Math F251 Final Exam Spring 2021

Name: Solutions

Section: ☐ **FXA (Sus)**
☐ **FXB (Maxwell)**
☐ **UX1 (Jurkowski)**

Rules:

You have 150 minutes (2.5 hours) to complete the exam.

Partial credit will be awarded, but you must show your work.

A scientific or graphing calculator (without symbolic manipulation) is allowed.

A one page sheet of paper (8 1/2 in. x 11 in.) with handwritten notes on one side is allowed.

No other aids are permitted.

Place a box around your $\boxed{\text{FINAL ANSWER}}$ to each question where appropriate.

Turn off anything that might go beep during the exam.

Academic Integrity Statement:

All students must affirm the following statements by initialing in the blanks provided. Students using their own paper must write out the statements in full.

____________ I will not seek or accept help from anyone.

____________ I will not use books, the internet or other disallowed aids.

____________ I understand correct answers without sufficient supporting work will be marked incorrect.

Problem	Possible	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Extra Credit	5	
Total	100	

1. (10 points)

Compute the derivatives of the following functions. You do not need to simplify your answers.

a. $f(t) = \ln(t)\arctan(2t)$

$$f'(t) = \frac{1}{t}\arctan(2t) + \frac{\ln(t)}{1+(2t)^2}\cdot 2$$

b. $g(x) = \dfrac{\cos(x^3)}{1-3x}$

$$g'(x) = \frac{-\sin(x^3)\cdot 3x^2\cdot(1-3x) + 3\cos(x^3)}{(1-3x)^2}$$

2. (10 points)

Compute the following indefinite integrals.

a. $\displaystyle\int \sin(1+x^4)x^3\,dx$

$u = 1+x^4 \quad du = 4x^3\,dx$

$$\int \sin(u)\frac{1}{4}\,du = -\frac{1}{4}\cos(u)$$

$$\boxed{-\frac{1}{4}(1+x^4)+C}$$

b. $\displaystyle\int \left(e^{-3x} + \frac{(\ln(x))^3}{x}\right)dx$

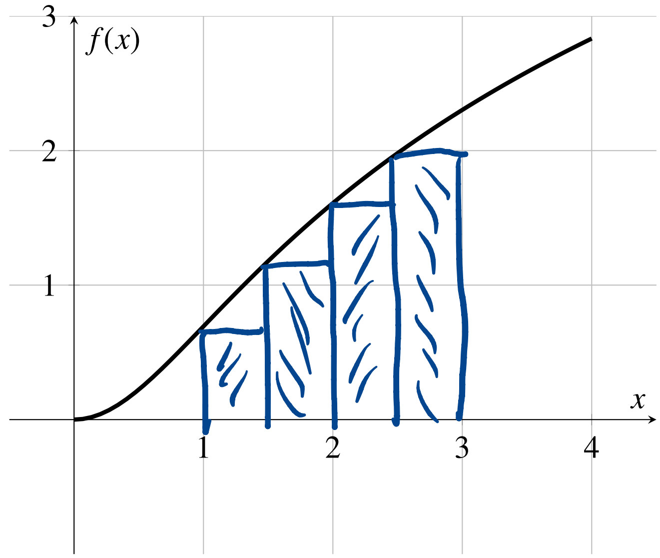
$u = \ln(x),\ du = \frac{1}{x}\,dx$

$$\int u^3\,du = \frac{u^4}{4}$$

$$\boxed{-\frac{1}{3}e^{-3x} + \frac{\ln(x)^4}{4} + C}$$

3. (10 points)

The graph of the function $f(x) = \ln(x^2 + 1)$ is shown.



a. On the graph sketch 4 rectangles, using **left** endpoints, that would be used to approximate

$$\int_1^3 \ln(x^2 + 1)\,dx.$$

b. Compute the approximation in part **(a)**. You do not need to simplify, but your answer should be in a form where a calculator would compute a numerical value.

$$\Delta x = \frac{3-1}{4} = \frac{1}{2}$$

$$L_4 = \frac{1}{2}\left[f(1) + f(3/2) + f(2) + f(5/2) \right]$$

$$= \frac{1}{2}\left[\ln(2) + \ln\left(\left(\tfrac{3}{2}\right)^2 + 1\right) + \ln(5) + \ln\left(\left(\tfrac{5}{2}\right)^2 + 1\right) \right]$$

4. (10 points)

A circular disk of metal is sitting in the sun and being heated. Its radius is increasing at a rate of 0.2 cm/hour because of thermal expansion. How fast is the area of the disk increasing when the radius of the disk is 40 cm? Units please!

$A = \pi r^2$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

$r = 40$

$\frac{dr}{dt} = 0.2$

$$\frac{dA}{dt} = 2\pi \cdot 40 \cdot 0.2$$

$$= 16\pi \ \text{cm}^2/\text{hour}$$

5. (10 points)

The temperature in a sauna in degrees Fahrenheit is given by

$$T(t) = 10\,t\,e^{-t/60}$$

where T is measured in degree Farenheit and $t \geq 0$ is measured in minutes.

a. What is the average rate of change of the temperature from time $t = 0$ to $t = 30$ minutes? Include units in your answer.

$$\frac{T(30)-T(0)}{30-0} = \frac{10 \cdot 30 e^{-1/2} - 0}{30 - 0}$$

$$= 10e^{-1/2} \approx 6.07 \text{ °F/min}$$

b. At what **rate** is the temperature changing at time $t = 0$? Include units in your answer.

$$T'(t) = 10e^{-t/60} + 10te^{-t/60} \cdot (-1/60)$$

$$T'(0) = 10e^0 + 1 = \boxed{10 \text{ °F/min}}$$

c. Compute $\lim_{t\to\infty} T(t)$ and explain what this number means in language the general public might understand.

$$\lim_{t\to\infty} 10te^{-t/60} = \lim_{t\to\infty} \frac{10t}{e^{t/60}}$$

$$\overset{\frac{\infty}{\infty}}{=} \lim_{t\to\infty} \frac{10}{\frac{1}{60}e^{t/60}} = 0$$

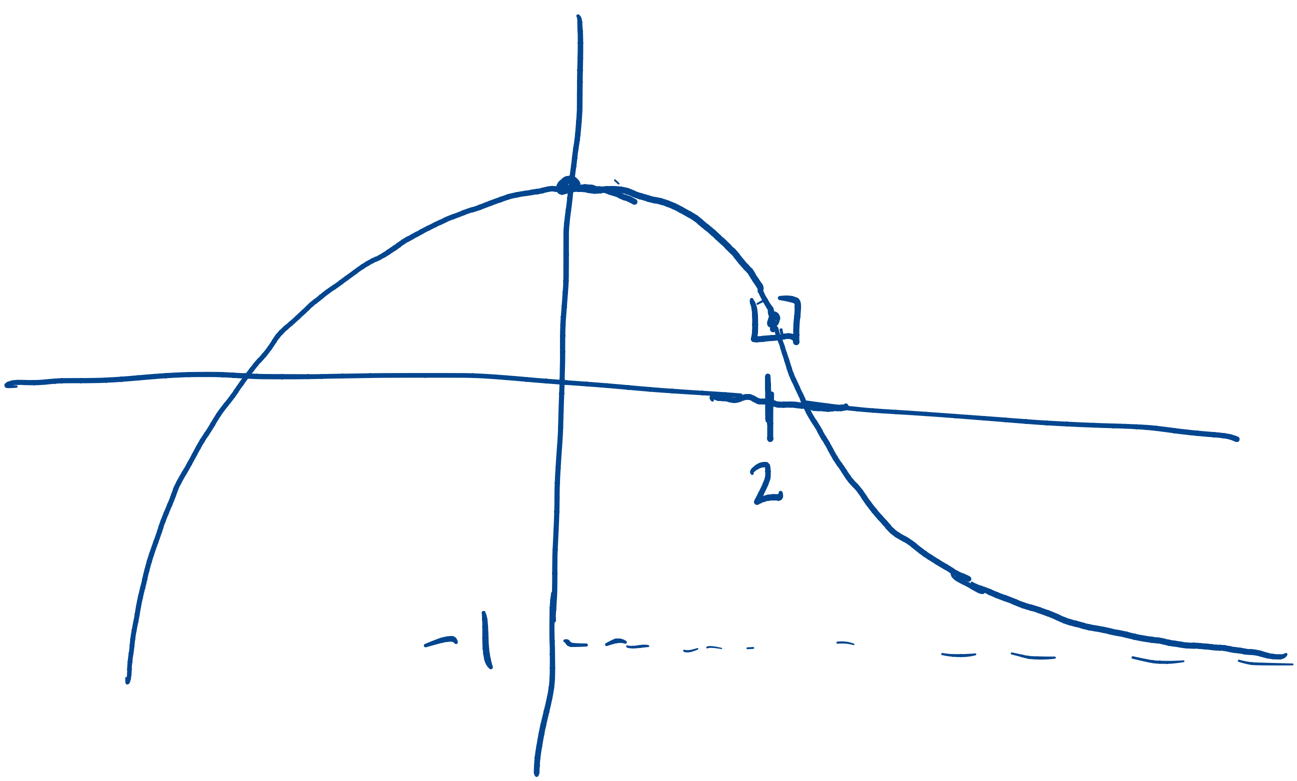
In the long term, the temp. in the sauna approaches 0 °F

6. (10 points)

The continuous function $G(x)$ with domain all real numbers has the properties below:

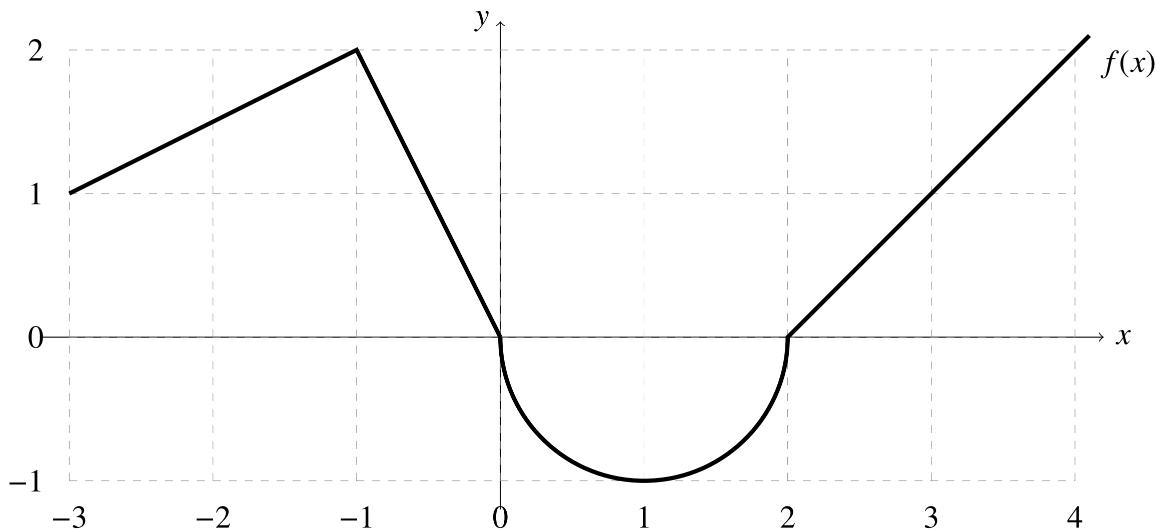
- $G(0) = 2$.
- $G'(x)$ is positive if $x < 0$.
- $G'(x)$ is negative if $x > 0$.
- $G''(x)$ is negative if $x < 2$
- $G''(x)$ is positive if $x > 2$.
- $\lim_{x\to\infty} G(x) = -1$.

Sketch the graph of $G(x)$. Your sketch should include the region $-4 \le x \le 4$ and should include a box at any points of inflection.



7. (10 points)

Consider the function $f(x)$ graphed below. Between $x = 0$ and 2, the graph is of a semicircle of radius 1.



a. What is the value of $f(3)$? 1

b. What is the value of $f'(-2)$? $1/2.$

c. What is the value of $\int_1^4 f(x)\,dx$? $2 + \pi/4$

d. At what values of x, if any, does $f'(x)$ not exist?

$-1,\ 0,\ 2$

The following questions concern $H(x) = \int_0^x f(s)\,ds$.

e. What is the value of $H(-3)$?

$$\int_0^{-3} f(s)\,ds = -\int_{-3}^{0} f(s)\,ds = -4$$

f. What is the value of $H'(1)$?

$$H'(1) = f(1) = -1$$

8. (10 points)

The **rate of change** of elevation of water in Turnagain Arm outside of Anchorage is given by

$$r(t) = 4\cos\left(\frac{\pi}{6}t\right)$$

meters per hour where t is measured in hours since midnight on a particular day.

a. Compute $r(0)$ and $r(3)$. Then explain what these numbers mean in language the general public would understand.

$r(0) = 4$ m/hour, $r(3) = 0$ m/hour.

At midnight, the tide is rising at 4 m/h, and at 3 am the tide is neither rising nor falling

b. Compute the net change in elevation of the water in meters from time $t = 0$ to time $t = 3$. Hint: Think about the Net Change Theorem!

$$\int_0^3 r(t)\,dt = \int_0^3 4\cos(\pi/6\, t)\,dt = \frac{24}{\pi}\sin\left(\frac{\pi}{6}t\right)\Big|_0^3$$

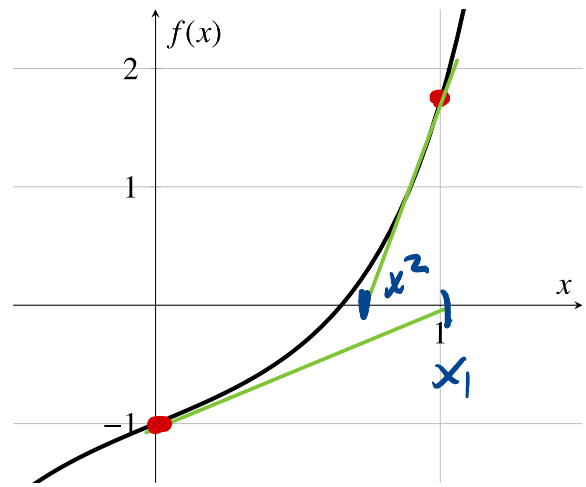
$$= \frac{24}{\pi} \approx 7.64 \text{ m}$$

c. At time $t = 0$, the elevation of water is 2 meters above sea level. What is the elevation of water above sea level at time $t = 3$?

9.64 m above sea level

9. (10 points)

The graph of the function $f(x) = xe^{x^2} - 1$ is shown.



a. Suppose Newton's method is used to find an approximate solution to $f(x) = 0$ from an **initial guess** of $x_1 = 0$. Sketch on the graph how the next approximations x_2 and x_3 will be found, labeling their locations on the x-axis.

b. Compute $f'(x)$.

$$f'(x) = e^{x^2} + 2x^2e^{x^2}$$

c. For $x_1 = 0$, compute the value of x_2 you illustrated in part **(a)**. You do not need to simplify, but your answer should be in a form where a calculator would compute a numerical value.

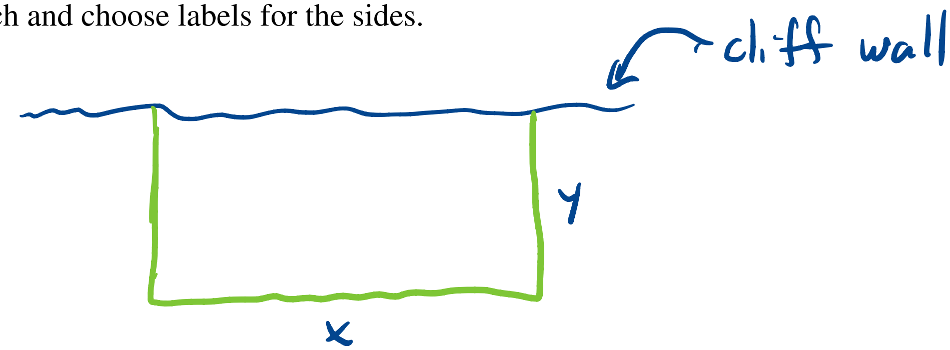
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0 - \frac{(-1)}{e^{0^2} + 2 \cdot 0} = \boxed{1}$$

10. (10 points)

A homesteader needs to fence off an area of 2000 m^2 for a grazing animal, and she figures building a rectangular area adjacent to the base of a cliff would work best. In other words, no fencing is needed along the cliff that forms one side of the area. What is the minimum amount of fencing needed?

a. Draw a sketch and choose labels for the sides.



b. Solve the problem. Indicate units in your answer.

$$A = xy \Rightarrow x = \frac{2000}{y}$$

perimeter: $P = 2y + x$

$$P = 2y + \frac{2000}{y}$$

$$P' = 2 - \frac{2000}{y^2} \qquad P' = 0 \Rightarrow y = \sqrt{1000}$$

$$P'' = \frac{2000}{y^3} \Rightarrow \text{concave up for } y > 0$$

$\Rightarrow$ only a global min is possible

$$x = \frac{2000}{y} = \frac{2000}{\sqrt{1000}} = 2\sqrt{1000}$$

Total fencing needed: $x + 2y = 4\sqrt{1000} \approx 126.5$ m

11. (Extra Credit: 5 points)

Find the derivative of

$$H(x) = \int_x^{x^2} e^{-s^2}\, ds.$$

$$H(x) = \int_x^c e^{-s^2} ds + \int_c^{x^2} e^{-s^2} ds$$

$$= -\int_c^x e^{-s^2} ds + \int_c^{x^2} e^{-s^2} ds$$

$$H'(x) = -e^{-x^2} + 2x e^{-x^4}$$