Math F251 — Midterm 1 — Spring 2022

Name: Solutions

Section:
- ☐ F01 (Jill Faudree)
- ☐ F02 (James Gossell)
- ☐ UX1 (James Gossell)

Rules:

You have 60 minutes to complete the exam.

Partial credit will be awarded, but you must show your work.

The exam is closed book and closed notes.

Calculators are not allowed.

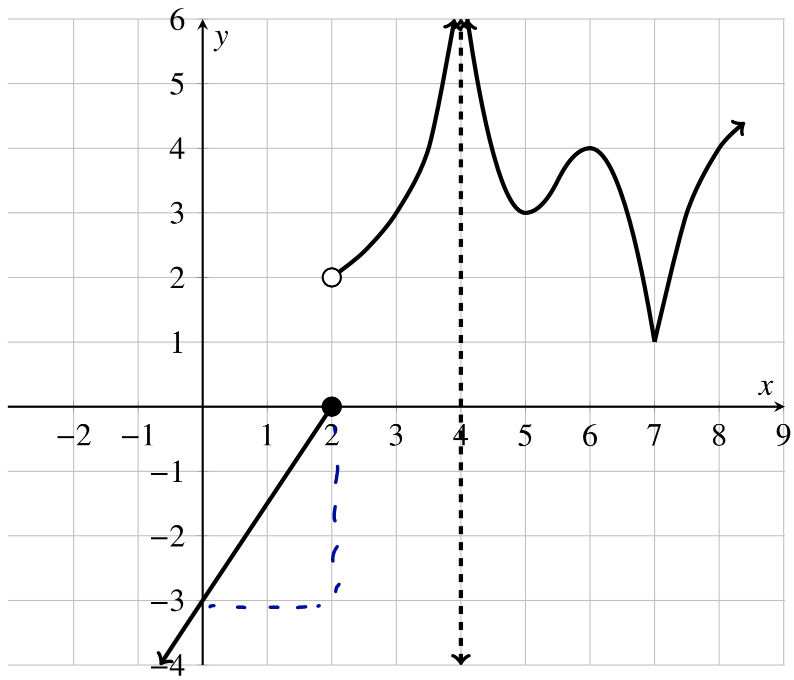
Place a box around your FINAL ANSWER to each question where appropriate.

Turn off anything that might go beep during the exam.

Good luck!

Problem	Possible	Score
1	12	
2	20	
3	10	
4	8	
5	6	
6	8	
7	8	
8	15	
9	15	
Extra Credit	5	
Total	100	

1. (12 points) Use the graph of the function $f(x)$, sketched below, to answer the questions. The dotted line at $x = 4$ represents a vertical asymptote. You must give the most complete answer; if an answer is ∞ or $-\infty$, you must indicate this.



(a) $\lim_{x\to 2^-} f(x) =$ 0

(b) $\lim_{x\to 2^+} f(x) =$ 2

(c) $\lim_{x\to 2} f(x) =$ DNE

(d) $f(2) =$ 0

(e) $\lim_{x\to 4} f(x) =$ $+\infty$

(f) $\lim_{x\to 7^+} f(x) =$ 1

(g) $f'(1) =$ 3/2

(h) $f'(5) =$ 0

(i) List the x-values where $f(x)$ fails to be continuous.

$x=2, x=4$

(j) List the x-values for which $f(x)$ fails to have a derivative.

$x=2, x=4, x=7$

2. (20 points) Evaluate the following limits. Give the most complete answer; if the limit is infinite, indicate that with ∞ or $-\infty$. If a value does not exist, write DNE.

(a) $\lim_{x\to 2^+} \frac{x-\sqrt{2+x}}{x^2+4} = \frac{2-\sqrt{4}}{4+4} = \frac{2-2}{8} = \frac{0}{8} = 0$

(b) $\lim_{x\to 10^+} \frac{1-4x}{(10-x)^3} = \cancel{\frac{-39}{0}} = -\infty$

as $x \to 10^+$ (#'s like 10.001), $(10-x)^3 \to 0^+$

and $1-4x \to -39$.

So the quotient is $\frac{-}{+} = -$

(c) $\lim_{x\to 1} \frac{x^2+2x-3}{2x^2-x-1} = \lim_{x\to 1} \frac{(x-1)(x+3)}{(x-1)(2x+1)} = \lim_{x\to 1} \frac{x+3}{2x+1} = \frac{1+3}{2\cdot 1+1} = \frac{4}{3}$

$1+2-3=0$

$2-1-1=0$

(factor + cancel!)

$\frac{1}{4}+\frac{1}{-4}=0$

(d) $\lim_{x\to -4} \left(\frac{\frac{1}{4}+\frac{1}{x}}{x+4}\right) = \lim_{x\to -4} \left(\frac{1}{x+4}\right)\left(\frac{x+4}{4x}\right) = \lim_{x\to -4} \frac{1}{4x} = \frac{1}{4(-4)} = \frac{-1}{16}$

$-4+4=0$

(algebra!)

3. (10 points) Find the derivative of $f(x) = x^2 + 3x$ using the limit definition of the derivative

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

No credit will be awarded for using other methods.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\overbrace{[(x+h)^2 + 3(x+h)]}^{f(x+h)} - \overbrace{[x^2+3x]}^{f(x)}}{h}$$

$$= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 + 3x + 3h - x^2 - 3x}{h}$$

$$= \lim_{h\to 0} \frac{2xh + h^2 + 3h}{h} = \lim_{h\to 0} \frac{h(2x+h+3)}{h}$$

$$= \lim_{h\to 0} (2x+h+3) = 2x+3$$

So $f'(x) = 2x+3$.

4. (8 points) Is the following function continuous at $x = 3$? Justify your answer. Answer: No. Not continuous.

$$f(x) = \begin{cases} \frac{6-2x}{x-3} & \text{if } x \le 3 \\ 2 & \text{if } x = 3 \\ x - 5 & \text{if } x \ge 3 \end{cases}$$

(check left limit) $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{6-2x}{x-3} = \lim_{x\to 3^-} \frac{-2(x-3)}{x-3} = \lim_{x\to 3^-} -2 = -2$

But $f(3) = 2$.

Since $\lim_{x\to 3^-} f(x) \neq f(2)$, the function isn't continuous at $x=3$.

(FYI: You could have checked the right side limit, too, and concluded the same thing...)

5. (6 points) Does the function $g(x) = 4\sin x - 5\cos x$ pass through the x-axis on the interval $[0, \frac{\pi}{2}]$? Justify your answer. (Hint: Use the Intermediate Value Theorem.)

- Observe that $g(x)$ is continuous, so the I.V. Thm applies.
- Check end points:

$$g(0) = 4\sin(0) - 5\cos(0) = 0 - 5 = -5 < 0$$
$$g(\tfrac{\pi}{2}) = 4\sin(\tfrac{\pi}{2}) - 5\cos(\tfrac{\pi}{2}) = 4 - 0 = 4 > 0$$

- Conclusion: Since $g(x)$ is below the x-axis at $x=0$, above the x-axis at $x = \frac{\pi}{2}$, and continuous, the I.V. Thm implies $g(x)$ must cross the x-axis somewhere in the interval $[0, \pi/2]$.

6. (8 points) Let $H(t)$ be the daily cost, in dollars, of heating a building when the outside temperature is t degrees Fahrenheit.

   (a) What is the meaning of $H(0) = 50$? Your answer should be a complete sentence and must include units.

   When the outside temperature is 0°F, it costs \$50 to heat the building for one day.

   (b) What are the units of $H'(t)$?

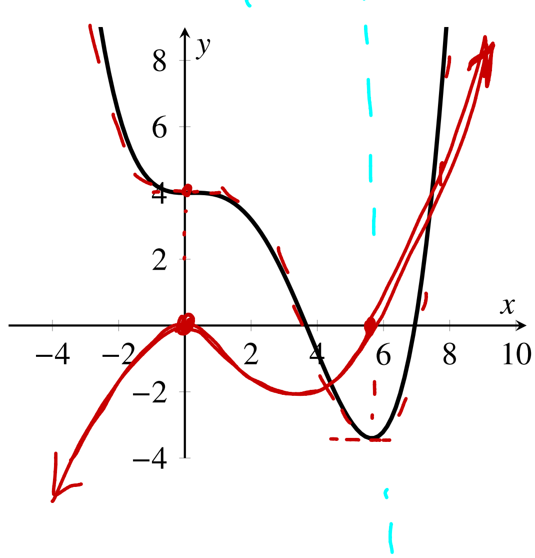
   \$ / °F

   (c) Do you expect $H'(t)$ to be positive or negative? **Justify** your answer using complete sentences.

   negative.
   As temperature (t) gets larger (warmer), we expect the heating costs to decrease.

7. (8 points) The graph of $g(x)$ is graphed below. On the same set of axes, sketch the graph of its derivative $g'(x)$.

– slope , + slope



8. (15 points) Find $f'(x)$ for each of the following expressions. You do not need to simplify.

   (a) $f(x) = 3\cos x + 2x^3 - \frac{1}{x^2} + \sqrt{5}$ $= 3\cos(x) + 2x^3 - x^{-2} + \sqrt{5}$

   $f'(x) = -3\sin(x) + 6x^2 + 2x^{-3}$

   product rule

   (b) $f(x) = x^{2.3} + x \cdot \cos(x)$

   $f'(x) = 2.3x^{1.3} + x(-\sin(x)) + 1 \cdot \cos(x)$

   $= 2.3x^{1.3} - x\sin(x) + \cos(x)$

   quotient rule

   (c) $f(x) = \frac{\sin x}{x^3 - 1}$

   $f'(x) = \dfrac{(x^3-1)(\cos(x)) - (\sin(x))(3x^2)}{(x^3-1)^2}$

   $= \dfrac{(x^3-1)\cos(x) - 3x^2\sin(x)}{(x^3-1)^2}$

9. (15 points) The mars rover launches a sensor vertically upward from an arm 1 meter above the surface of the planet. The height, in meters, of the sensor t seconds after launch is given by

$$h(t) = 1 + 8t - 2t^2.$$

Include units in your answers.

(a) Find the velocity function and the acceleration function for the sensor.

$h'(t) = v(t) = 8 - 4t$ m/s

$h''(t) = a(t) = -4$ m/s^2

(b) What is the initial velocity of the sensor?

initial velocity means $v(0)$.

$v(0) = 8 - 4 \cdot 0 = 8$ m/s

aside:

$h(4) = 1 + 8 \cdot 4 - 2 \cdot 4^2$
$= 33 - 32 = 1$

$h(1) = 1 + 8 \cdot 1 - 2 \cdot 1 = 9 - 2 = 7$

(c) What is the average velocity of the sensor in the interval from $t = 1$ to $t = 4$?

average velocity $= \dfrac{\Delta h}{\Delta t} = \dfrac{h(4) - h(1)}{4 - 1} = \dfrac{1 - 7}{4 - 1} = \dfrac{-6}{3} = -2$ m/s

(d) Four seconds after launch (when $t = 4$) is the sensor speeding up or slowing down? Justify your answer.

answer: Speeding up.

Justification: At $t = 4$, $v(4) = -8$ and $a(4) = -4$. Since both are negative, the sensor is speeding up.

(e) What is the acceleration due to gravity on mars?

4 m/s^2

10. **Extra Credit** (5 points) Find $\frac{d}{dx}(\sec x)$ by rewriting $\sec x$ as $\frac{1}{\cos x}$ and using the quotient rule. For full credit, please simplify your final answer.

$$\frac{d}{dx}\left[\sec x\right] = \frac{d}{dx}\left[\frac{1}{\cos x}\right] = \frac{(\cos x)(0) - 1\cdot(-\sin x)}{\cos^2 x}$$

$$= \frac{\sin x}{\cos^2 x} = \frac{\sin x}{\cos x}\cdot\frac{1}{\cos x} = \tan x \sec x$$