Math F251 Midterm 2 Spring 2022

Name: Solutions

Section:
- ☐ 001 (Jill Faudree)
- ☐ 002 (James Gossell)
- ☐ 005 (James Gossell)

Rules:

You have 60 minutes to complete the exam.

Partial credit will be awarded, but you must show your work.

The exam is closed book and closed notes.

Calculators are not allowed.

Place a box around your FINAL ANSWER to each question where appropriate.

Turn off anything that might go beep during the exam.

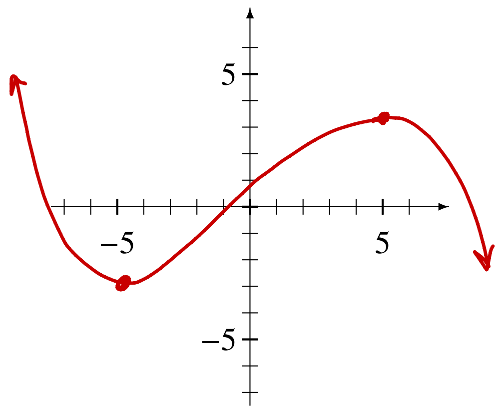
Good luck!

Problem	Possible	Score
1	12	
2	10	
3	10	
4	10	
5	10	
6	10	
7	16	
8	12	
9	10	
Extra Credit	5	
Total	100	

1. (12 points) Sketch graphs which satisfy the given conditions. **There are many correct answers.**

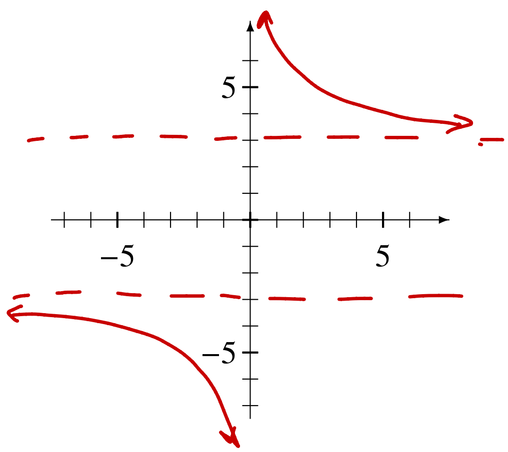
(a) Sketch a graph of a function $f(x)$ that has

- a local minimum at $x = -5$, and
- a local maximum at $x = 5$.



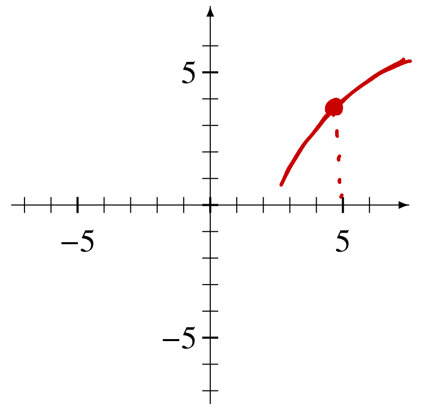
(b) Sketch a graph of a function $g(x)$ such that

- $\lim_{x\to\infty} g(x) = 3$, and
- $\lim_{x\to-\infty} g(x) = -3$.



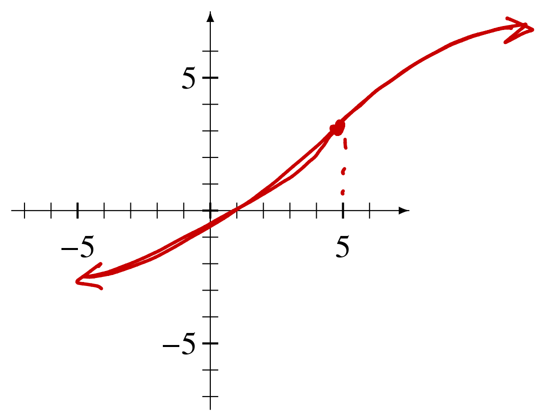
(c) Sketch a graph of a function $h(x)$ such that

- $h'(5) > 0$, and increasing
- $h''(5) < 0$. concave down



(d) Sketch a graph of a function $k(x)$ that has

- $k''(x) > 0$ for $x < 5$, and
- $k''(x) < 0$ for $x > 5$.



2. (10 points) Evaluate the following limits. You must show your work to earn full credit. If you apply L'Hopital's Rule, you should indicate this.

(a) $\lim_{x\to 1} \dfrac{6x^2 - 12x + 6}{-x^4 + x^3 + x^2 - x}$ $\overset{H}{=} \lim_{x\to 1} \dfrac{12x-12}{-4x^3+3x^2+2x-1} \overset{H}{=} \lim_{x\to 1} \dfrac{12}{-12x^2+6x+2}$

plugin $= \dfrac{6-12+6}{-1+1+1-1} = \dfrac{0}{0}$

plug in to get $\dfrac{12-12}{-4+3+2-1} = \dfrac{0}{0}$

$= \dfrac{12}{-12+6+2} = \dfrac{12}{-4}$

$= -3$

(b) $\lim_{x\to 0} \dfrac{x\cos 2x}{\sin 2x}$ $\overset{H}{=} \lim_{x\to 0} \dfrac{\cos(2x) - 2x\sin(2x)}{2\cos(2x)} = \dfrac{1-0}{2\cdot 1} = \dfrac{1}{2}$

form $\frac{0}{0}$

3. (10 points)

(a) Find the linear approximation of the function $f(x) = x^3$ when $a = 2$.

$f(2) = 2^3 = 8$

$f'(x) = 3x^2$

$f'(2) = 3\cdot 2^2 = 12$

$L(x) = 8 + 12(x-2)$

(b) Use the linear approximation to estimate $(2.02)^3$. Your answer must be in the form of a decimal.

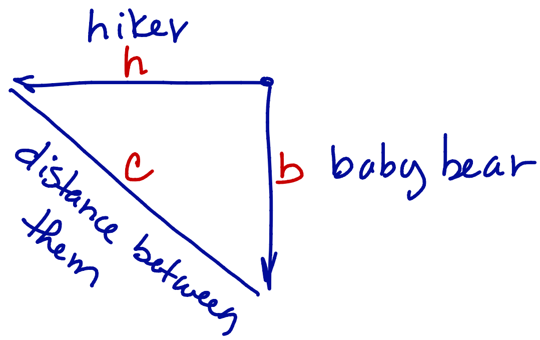
$L(2.02) = 8 + 12(2.02-2) = 8 + 12(0.02)$

$= 8 + 0.24$

$= 8.24$

4. (10 points) **[Related Rate Problem]** A hiker and a baby bear meet in the woods. Both are terrified! The hiker runs west at 3 meters per second. The bear lumbers south at 4 meters per second.

   (a) Draw a diagram.



   (b) Determine how fast the distance between the hiker and bear is increasing 2 seconds after the hiker and baby bear meet. Include units with your answer.

List of information

$\frac{dh}{dt} = 3$ m/s

$\frac{db}{dt} = 4$ m/s

want $\frac{dc}{dt}$

when $t=2$:

$h = 6$

$b = 8$

$c = \sqrt{6^2+8^2}$

$= 10$

$$c^2 = h^2 + b^2$$

$$2c\frac{dc}{dt} = 2h\frac{dh}{dt} + 2b\frac{db}{dt}$$

$$\frac{dc}{dt} = \frac{h\frac{dh}{dt} + b\frac{db}{dt}}{c}$$

$$= \frac{6\cdot 3 + 8\cdot 4}{10} = \frac{18+32}{10}$$

$$= \frac{50}{10} = 5 \text{ m/s}$$

5. (10 points)

(a) Find all critical values of $f(x) = \dfrac{x^2+2}{2x+1}$ on the interval $[0, 4]$.

$$f'(x) = \frac{(2x+1)(2x) - (x^2+2)(2)}{(2x+1)^2} = \frac{4x^2+2x-2x^2-4}{(2x+1)^2} = \frac{2x^2+2x-4}{(2x+1)^2}$$

$$= \frac{2(x^2+x-2)}{(2x+1)^2} = \frac{2(x+2)(x-1)}{(2x+1)^2}$$

$f'(x)=0$ when $x=-2$ or $x=1$; $f'(x)$ undefined at $x=-\frac{1}{2}$.

Only $x=1$ is in the interval.

(b) Identify the absolute maximum and absolute minimum of the function $f(x)$ on the interval $[0, 4]$.

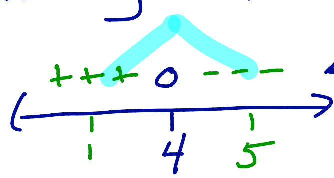
table of values

x	0	1	4
$f(x)$	$\frac{2}{1}=2$	$\frac{1+2}{2+1}=1$	$\frac{16+2}{2\cdot4+1}=\frac{18}{9}=2$

$\boxed{\textbf{Absolute max: } y = 2 \qquad \textbf{Absolute min: } y = 1}$

6. (10 points) **[Optimization Problem]** Find the two **positive** numbers x and y on the circle $x^2+y^2 = 48$ that maximize the product $P = xy^2$.
   Note: Your solutions must use Calculus to **justify** that your answer is correct.

- maximize $P = xy^2$
- write as a function of 1 variable: $P(x) = x(48-x^2) = 48x - x^3$
- find derivative: $P'(x) = 48 - 3x^2$
- set = to 0: $48-3x^2=0$ or $3x^2=48$ or $x^2=16$
  So $x=\pm 4$. But only $x=4$ is positive.
- Justify: $\boxed{\text{opt1}}$ 1st derivative test



←Sign of P'

a max at $x=4$

$\boxed{\text{opt2}}$ 2nd derivative test.
$P''(x) = -6x$, $P''(4) = -6(4) < 0$.
a max at $x=4$.

- Find y: $y^2 = 48 - x^2 = 48 - 16 = 32$.
  So $y = \pm\sqrt{32} = \pm\sqrt{16\cdot 2} = \pm 4\sqrt{2}$
  We only want positive y-values so $y = 4\sqrt{2}$

$\boxed{x = 4 \qquad y = 4\sqrt{2}}$

7. (16 points) Use the information below to answer questions about the function $f(x)$. You must show your work to earn full credit.

$$f(x) = x^{4/3} - 4x^{1/3}, \quad f'(x) = \frac{4x-4}{3x^{2/3}}, \quad f''(x) = \frac{4x+8}{9x^{5/3}}.$$

(a) Determine the intervals on which $f(x)$ is increasing/decreasing.

$f' = 0$ when $x = 1$

f' undef when $x = 0$

f is ↑ on $(1, \infty)$

f is ↓ on $(-\infty, 1)$

− − − − − − 0 + + + ← sign of f'

$-1 \quad 0 \quad \frac{1}{2} \quad 1 \quad 2$

(b) Find the x-values that correspond to any local maximums or local minimus of $f(x)$.

f has a local min at $x = 1$.

f has no local max.

(c) Find the intervals on which $f(x)$ is concave up and concave down.

$f'' = 0$ when $x = -2$

f'' undef. when $x = 0$

f is ccup on $(-\infty, -2) \cup (0, \infty)$

ccdown on $(-2, 0)$

+ + + 0 − − − + + + ← sign of f''

$-3 \quad -2 \quad -1 \quad 0 \quad 1$

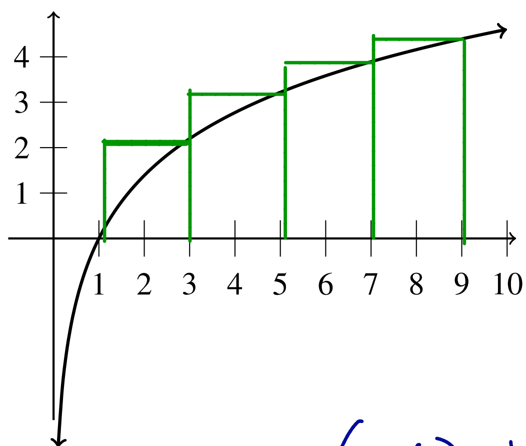
(d) Find the x-values of any inflection points of $f(x)$. If there aren't any, you must explicitly state this and justify your answer.

f has inflection points at $x = -2$ and $x = 0$.

The convavity of $f(x)$ changes on either side of $x = -2$ and $x = 0$. (See above)

8. (12 points)

(a) Estimate the area under the curve $f(x) = \ln(x)$ on the interval $[1, 9]$ using R_4. (That is, use 4 approximating rectangles and right-hand end points.) Sketch the approximating rectangles on the graph below and estimate the sum of the areas.
Note: You are obviously not expected to compute things like ln(3). It is acceptable to have numbers like this in your final answer.



$A \approx 2\left(\ln(3)+\ln(5)+\ln(7)+\ln(9)\right)$

(b) In fact, the area under the graph $f(x)$ on the interval $[1, 9]$ about 11.8. Use this fact to evaluate the definite integrals below:

i. $\int_1^9 2\ln(x)\,dx = 2(11.8) = 23.6$

ii. $\int_1^9 2 + \ln(x)\,dx = 2\cdot 8 + 11.8 = 16 + 11.8 = 27.8$

(height)·(base) = area of rectangle.

or

$\int_1^9 (2+\ln(x))dx = \int_1^9 2\,dx + \int_1^9 \ln(x)dx = 2(9-1) + 11.8$

$= 27.8$

9. (10 points) Evaluate the indefinite integrals below.

(a) $\int (4\cos(x) + x^{2.3} + 10)\,dx$ $= 4\sin(x) + \frac{x^{3.3}}{3.3} + e^2 x + C$

(b) $\int (x^2+4)^2\,dx$ $= \int (x^4 + 8x^2 + 16)\,dx = \frac{1}{5}x^5 + \frac{8}{3}x^3 + 16x + C$

Extra Credit (5 points): The function $P(t) = \dfrac{5e^t}{e^t+2}$ models a population of caribou (in hundreds) over time t in years. Evaluate the $\lim_{x\to\infty} P(t)$ and interpret the answer in the context of the problem. (Your interpretation should be a complete sentence that a regular person can understand.)

$$\lim_{t\to\infty} \frac{5e^t}{e^t+2} \cdot \frac{\frac{1}{e^t}}{\frac{1}{e^t}} = \lim_{t\to\infty} \frac{5}{1+2e^{-t}} = \frac{5}{1+0} = 5$$

Interpretation:
This model indicates that in the long-term, the population of caribou will stabilize at 500 individuals.