Math F251 — Final Exam — Spring 2023

Name: Solutions

Rules:

You have 2 hours to complete the final exam.

Partial credit will be awarded, but you must show your work.

You may have a single handwritten 3×5 notecard.

Calculators are not allowed.

Place a box around your $\boxed{\text{FINAL ANSWER}}$ to each question where appropriate.

Turn off anything that might go beep during the exam.

Good luck!

Problem	Possible	Score
1	8	
2	8	
3	8	
4	10	
5	12	
6	10	
7	~~10~~ 12	
8	12	
9	12	
10	8	
Extra Credit	5	
Total	100	

1. (8 points) Evaluate the limits. An answer without clear, mathematically precise work, will not earn full credit. Any use of L'Hôpital's Rule should be indicated using an **H** over the equal sign.

   (a) $\lim_{x\to 1} \frac{x^2-1}{\cos(\frac{\pi}{2}x)} \overset{H}{=} \lim_{x\to 1} \frac{2x}{-\frac{\pi}{2}\sin(\frac{\pi}{2}x)} = \frac{2}{-\frac{\pi}{2}} = \frac{-4}{\pi}$

   form $\frac{0}{0}$

   (b) $\lim_{x\to -\infty} \frac{(1-x^3)\cdot 1/x^3}{(2x^3+4x^2-9)\cdot 1/x^3} = \lim_{x\to -\infty} \frac{\frac{1}{x^3}-1}{2+\frac{4}{x}-\frac{9}{x^3}} = \frac{-1}{2}$

2. (8 points) Find the derivative. You do not need to simplify your answer.

   (a) $B(x) = (1+\sqrt{x})\ln(5x^2+x)$

   $B'(x) = \frac{1}{2}x^{-\frac{1}{2}}\ln(5x^2+x) + (1+\sqrt{x})\cdot\left(\frac{10x+1}{5x^2+x}\right)$

   (b) $\cos(2x) + xe^y = 4y^3$ (Find $\frac{dy}{dx}$.)

   $-2\sin(2x) + 1\cdot e^y + xe^y\frac{dy}{dx} = 12y^2\frac{dy}{dx}$

   $-2\sin(2x) + e^y = \frac{dy}{dx}(12y^2 - xe^y)$

   $\frac{dy}{dx} = \frac{-2\sin(2x)+e^y}{12y^2 - xe^y}$

3. (8 points) Evaluate the integrals. You do not need to simplify your answer.

(a) $\displaystyle\int (3\sec^2(\theta) + \frac{6}{\theta} + \ln(2))\, d\theta$

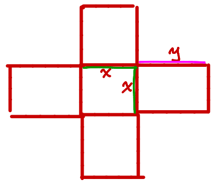
$$= 3\tan(\theta) + 6\ln|\theta| + \theta\ln(2) + C$$

(b) $\displaystyle\int \frac{2}{1+\frac{4}{9}t^2}\, dt$ $\displaystyle = 2\int \frac{dt}{1+\left(\frac{2}{3}t\right)^2}$

$$= 2\cdot\frac{3}{2}\arctan\left(\frac{2}{3}t\right) + C$$

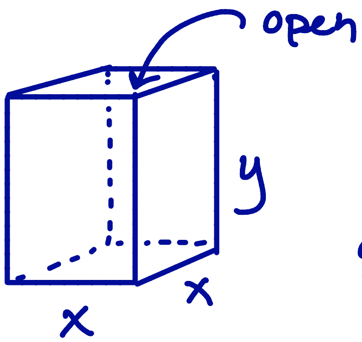
$$= 3\arctan\left(\frac{2}{3}t\right) + C$$

alt. picture



4. (10 points) An box with a square base and an open top has volume 4000 cm^3. What dimensions of the box will minimize its surface area?
You must show your work and use calculus to justify your answer.

(a) Draw and label a diagram. Then write an equation for the surface area of the box in terms of a single variable.



$V = x^2 y = 4000 \text{ cm}^3$

So $y = 4000 x^{-2}$

$S = x^2 + 4xy = x^2 + 4x(4000 x^{-2})$

So $S(x) = x^2 + 16000 x^{-1}$ on $(0, \infty)$

(b) Use Calculus to find the dimensions of the box that minimize the surface area.

$S'(x) = 2x - 16000 x^{-2} = 0$

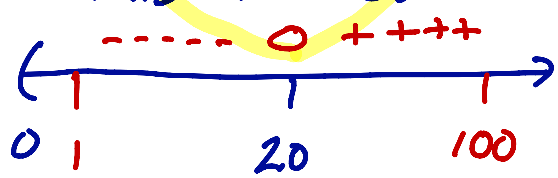
$2x = \frac{16000}{x^2}$

$x^3 = 8000$

$x = 20 \text{ cm}$

$y = \frac{4000}{(20)^2} = \frac{4000}{400} = 10 \text{cm}$

First Der. Test



$S'(1) = 2 - 16000 < 0$

$S'(100) = 200 - \frac{16,000}{10,000} > 0$

So S has a minimum at $x = 20\text{cm}$, $y = 10\text{cm}$

5. (12 points) The velocity of an object is given by $v(t) = \dfrac{t}{t^2+1}$ on the interval $[0, \infty)$, where v is measured in meters per second and t is measured in seconds.

   (a) Find an expression for $s(t)$, the position of the particle at time t, if $s = 1$ when $t = 0$.

$$s(t) = \int \frac{t}{t^2+1}\,dt = \frac{1}{2}\ln(t^2+1) + C.$$

when $t=0$, $1 = s(0) = \frac{1}{2}\ln(1) + C = 0 + C$. So $C = 1$.

$$s(t) = 1 + \ln(t^2+1)$$

   (b) Find an expression for $a(t)$, the acceleration of the object at time t.

$$a(t) = \frac{(t^2+1)(1) - t(2t)}{(t^2+1)^2} = \frac{t^2+1-2t^2}{(t^2+1)^2} = \frac{1-t^2}{(t^2+1)^2}$$

   (c) Determine at what time, t, is the velocity of the particle maximized. Use Calculus to show that your answer is correct..

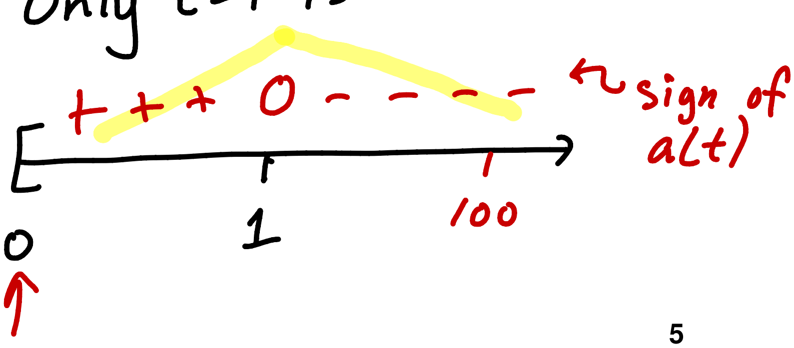
maximize $v(t)$.

Observe $a(t) = 0$ when $t = \pm 1$.

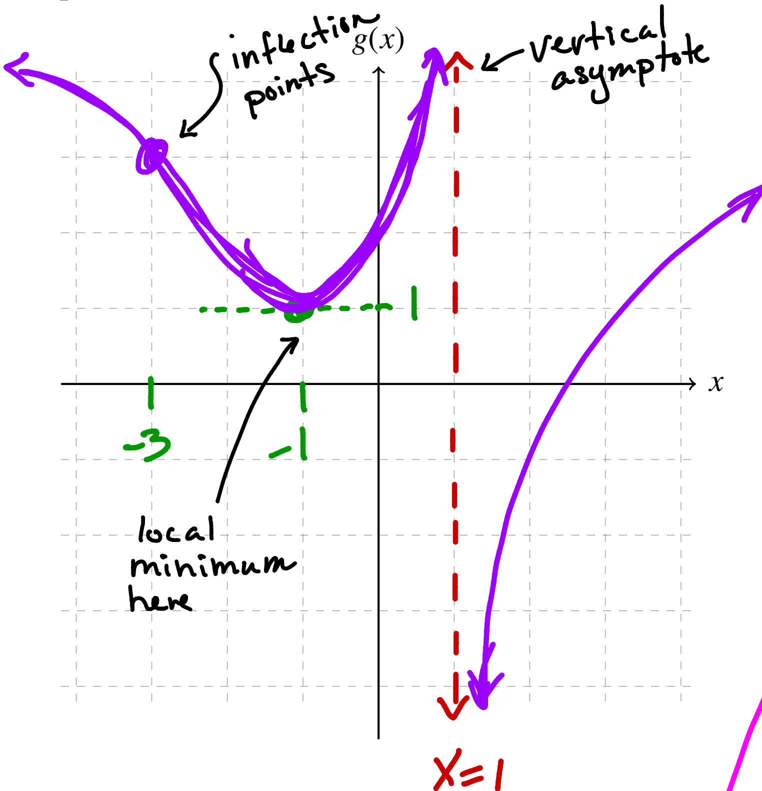
Only $t = 1$ is in domain.

$a(0) > 0$, $a(100) < 0$

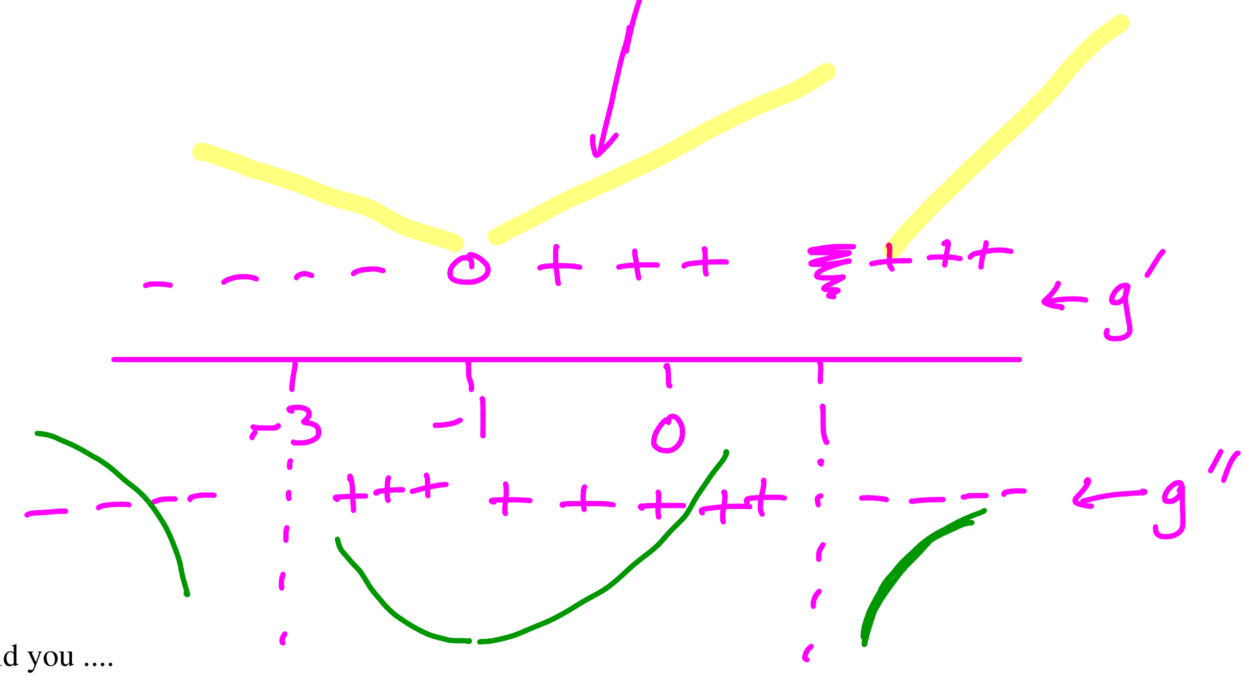
Velocity is maximized when $t = 1$.



6. (10 points) Use the axes below to sketch a graph of a function $g(x)$ that satisfies **all** of the conditions in the bulleted list. Make sure to label any asymptotes, minimums or maximums, and inflection points. (See check list.)



- $g(x)$ is continuous on its domain $(-\infty, 1) \cup (1, \infty)$.
- $g(-1) = 1$, $g'(-1) = 0$ ← hor. tangent
- $\lim_{x\to 1^-} g(x) = \infty$, $\lim_{x\to 1^+} g(x) = -\infty$ v.a. $x=1$
- $g'(x) > 0$ on $(-1, 1) \cup (1, \infty)$
- $g'(x) < 0$ on $(-\infty, -1)$
- $g''(x) > 0$ on $(-3, 1)$
- $g''(x) < 0$ on $(-\infty, -3) \cup (1, \infty)$



Did you ....

☐ label any asymptotes with its equation?

☐ label any maximums or minimums with **local min, local max, absolute min,** or **absolute max**?

☐ label any inflection points with **inflection point**?

7. (12 points) The **rate of change** of the volume of water in a tank is given by

$$r(t) = \frac{1}{2}t - 5$$

where r is measured in liters per minute and t is measured in minutes since the monitoring began.

(a) Compute $r(0)$ and $r(30)$. Then explain what these numbers mean in language the general public would understand.

$r(0) = -5$ L/min ; $r(30) = 15 - 5 = 10$ L/min

When monitoring begins, the tank is losing water at a rate of 5 L per minute. Thirty minutes later, the tank is gaining water at a rate of 10 L per minute.

(b) Compute the net change in the volume of water in liters from time $t = 0$ to time $t = 10$.

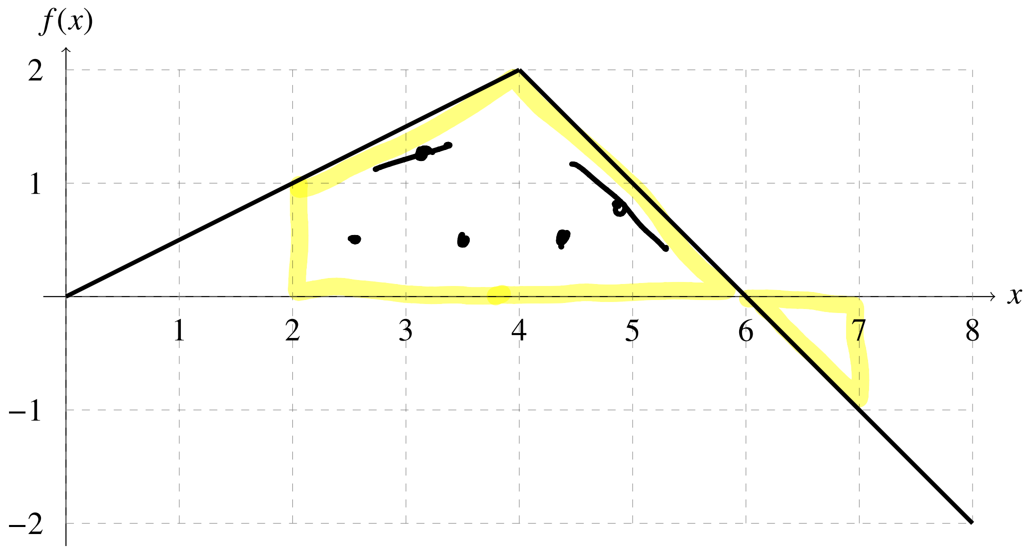
$$\text{net change} = \int_0^{10} \left(\frac{1}{2}t - 5\right) dt = \frac{1}{4}t^2 - 5t \Big]_0^{10}$$

$$= \frac{1}{4}(10)^2 - 5(10) - [0] = 25 - 50 = -25\text{ L}$$

(c) At time $t = 0$, the tank contains 200 liters of water. What is the volume of water in the tank at time $t = 10$?

Volume at $t = 10$ $= 200 - 25 = 175$ L

8. (12 points) Consider the function $f(x)$ with domain $[-3, 4]$ graphed below.



(a) What is the value of $f(4)$? 2

(b) What is the value of $f'(2)$? $\frac{1}{2}$

(c) Evaluate $\int_2^7 f(x)\,dx$. $= \left(2+1+1+1\right) - \left(\frac{1}{2}\right) = 4.5$

The following questions concern $A(x) = \int_0^x f(s)\,ds$.

(d) What is the value of $A(4)$? $= \int_0^4 f(s)\,ds = \frac{1}{2}\cdot 4 \cdot 2 = 4$

(e) What is the value of $A'(6)$. 0

(f) Does $A(x)$ have a maximum? Explain your answer.

Yes at $x=6$, $A'=0$ and A' changes from positive to negative

9. (12 points) The temperature of a cup of coffee is modeled by the function

$$f(t) = 110e^{-t/10} + 40$$

where f is measured in degrees Fahrenheit and t is measured in minutes after the coffee was poured into the cup.

(a) Compute $f'(0)$. Then explain what this number means in language the general public could understand.

$$f'(t) = 110 \cdot \left(-\frac{1}{10}\right) e^{-t/10}$$

$$f'(0) = -11 \ {}^\circ F/min$$

The temperature of the coffee is decreasing at a rate of 11 °F per minute when the coffee is first poured.

(b) Compute $\lim_{t\to\infty} f(t)$. Then explain what this number means in language the general public could understand.

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} \frac{110}{e^{t/10}} + 40 = 40$$

In the long run, the temperature of the coffee approaches 40 °F.

10. (8 points) The radius of a spherical balloon is increasing at a rate of 2 cm/s. At what rate is the surface area of the balloon changing when the radius of the balloon is 5 cm? (Note that the surface area of a sphere is given by $SA = 4\pi r^2$.) Include units with your answer.

$\frac{dr}{dt} = 2$

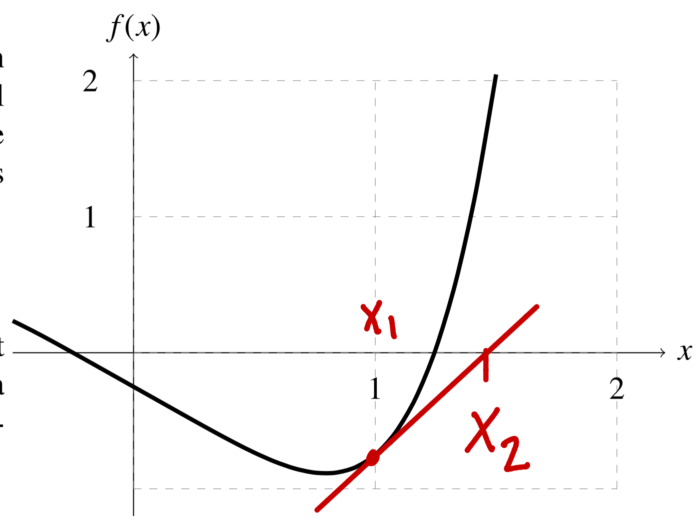
Find $\frac{dSA}{dt}$ when $r=5$

$$\frac{dSA}{dt} = 8\pi r \frac{dr}{dt} = 8\pi(5)(2)$$

$$= 80\pi \ cm^2/s$$

Extra Credit (5 points) The graph of the function $f(x) = \frac{1}{2}x^5 - x - \frac{1}{4}$ is shown.

a. Suppose Newton's method is used to find an approximate solution to $f(x) = 0$ from an initial guess of $x_1 = 1$. Sketch on the graph how the next approximation x_2 will be found, labeling its location on the x-axis.



b. For $x_1 = 1$, give a formula for x_2. You do not need to simplify, but your answer should be in a form where a calculator would compute a numerical value.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= x_1 - \frac{\frac{1}{2}x_1^5 - x_1 - \frac{1}{4}}{\frac{5}{2}x_1^4 - 1} = 1 - \left(\frac{\overbrace{\frac{1}{2} - 1 - \frac{1}{4}}^{-\frac{3}{4}}}{\underbrace{\frac{5}{2} - 1}_{3/2}}\right)$$

at $x_1 = 1$

$= 1.5$