Math F251 — Midterm 1 — Spring 2023

Name: Solutions

Rules:

You have 90 minutes to complete the exam.

Partial credit will be awarded, but you must show your work.

You may have a single handwritten 3×5 notecard.

Calculators are not allowed.

Place a box around your FINAL ANSWER to each question where appropriate.

Turn off anything that might go beep during the exam.

Good luck!

Problem	Possible	Score
1	12	
2	8	
3	10	
4	20	
5	12	
6	15	
7	6	
8	17	
Extra Credit	5	
Total	100	

1. (12 points)

   (a) State the definition of $f'(x)$, the derivative of the function $f(x)$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

   (b) Find the derivative of $f(x) = 4 - \sqrt{x}$ using the limit definition of the derivative. No credit will be awarded for using other methods.

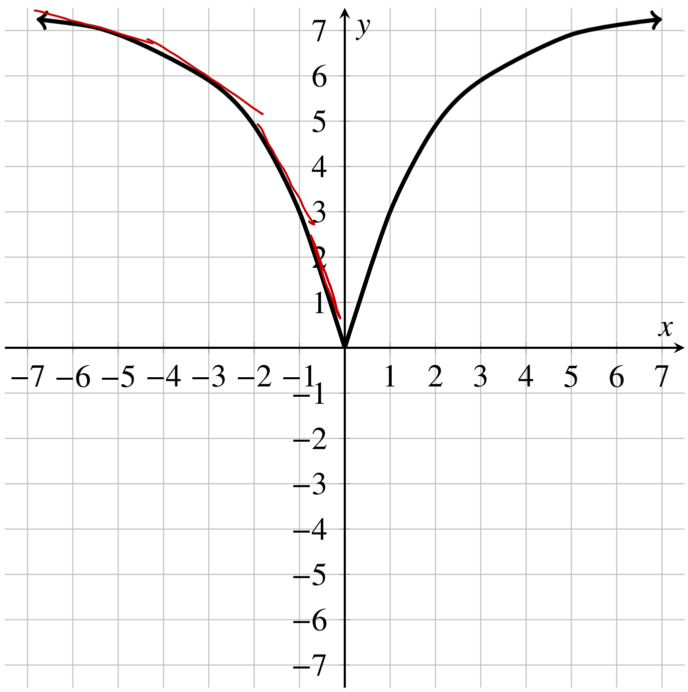
$$f'(x) = \lim_{h\to 0} \frac{(4-\sqrt{x+h})-(4-\sqrt{x})}{h}$$

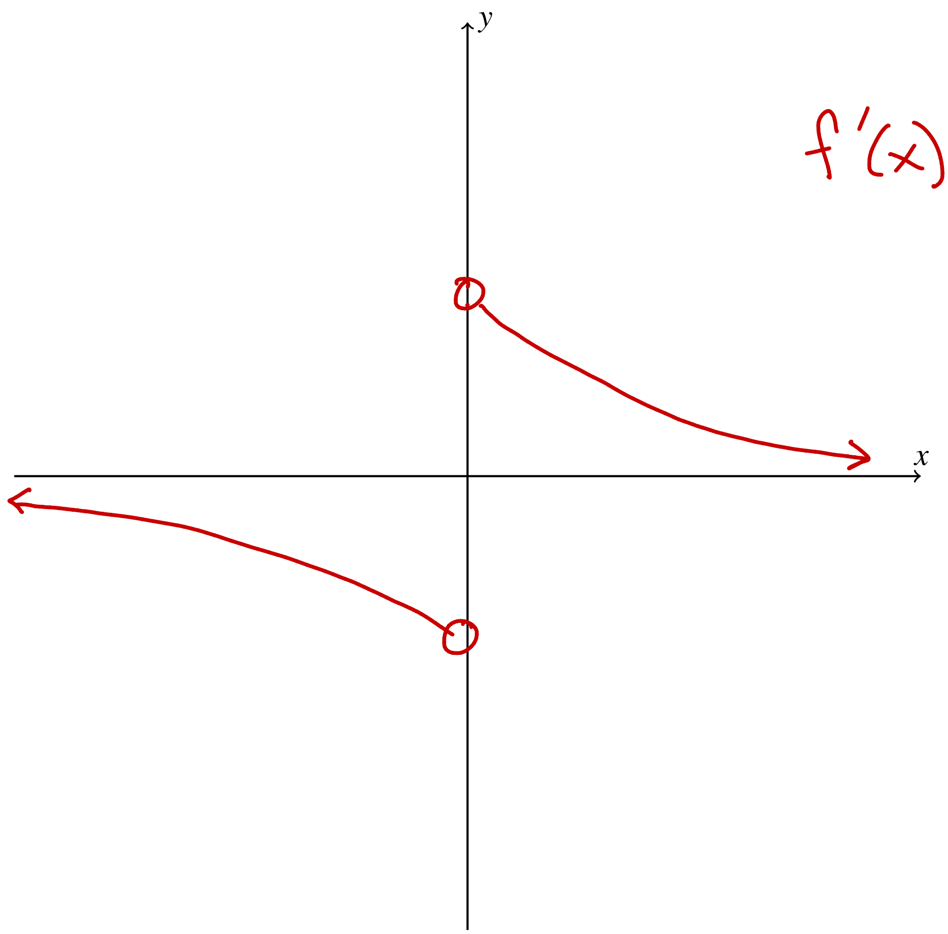
$$= \lim_{h\to 0} \frac{\sqrt{x}-\sqrt{x+h}}{h} \cdot \frac{(\sqrt{x}+\sqrt{x+h})}{(\sqrt{x}+\sqrt{x+h})}$$

$$= \lim_{h\to 0} \frac{x-(x+h)}{h(\sqrt{x}+\sqrt{x+h})} = \lim_{h\to 0} \frac{-h}{h(\sqrt{x}+\sqrt{x+h})}$$

$$= \lim_{h\to 0} \frac{-1}{\sqrt{x}+\sqrt{x+h}} = \frac{-1}{2\sqrt{x}}$$

2. (8 points) Use the graph of the function $g(x)$, drawn in the figure below, to sketch the graph of $g'(x)$ on the set of axes below.





3. (10 points) Let $f(x) = \begin{cases} \frac{\sqrt{x}+5}{2} & x < 9 \\ 6 & x = 9 \\ 3 + e^{x-9} & 9 < x \end{cases}$.

(a) Show that $\lim_{x\to 9} f(x)$ exists.

$$\lim_{x\to 9^-} f(x) = \lim_{x\to 9^-} \frac{\sqrt{x}+5}{2} = \frac{\sqrt{9}+5}{2} = \frac{3+5}{2} = 4$$

$$\lim_{x\to 9^+} f(x) = \lim_{x\to 9^+} 3+e^{x-9} = 3+e^0 = 3+1 = 4$$

Since the left and right side limits exist and are equal, the two-sided limit exists.

(b) Determine if $f(x)$ is continuous at $x = 9$ and justify your answer with a mathematical ~~equation~~ expression.

$f(x)$ is not continuous at $x=9$ because

$$\lim_{x\to 9} f(x) = 4 \neq 6 = f(9).$$

4. (20 points) Evaluate the following limits. Show your work to earn full credit. Be careful to use proper notation.

(a) $\lim_{x\to c} \frac{\frac{1}{c}-\frac{1}{x}}{x-c} = \lim_{x\to c} \frac{\frac{x-c}{cx}}{x-c} = \lim_{x\to c} \frac{1}{cx} = \frac{1}{c^2}$

(b) $\lim_{x\to 4^-} \frac{\sqrt{x}}{x^2-2} = \frac{\sqrt{4}}{4^2-2} = \frac{2}{16-2} = \frac{2}{14} = \frac{1}{7}$

(c) $\underbrace{\lim_{x\to 3} \frac{x^2-9}{2x^2-5x-3}}_{\text{form } \frac{0}{0}} = \lim_{x\to 3} \frac{(x-3)(x+3)}{(x-3)(2x+1)} = \lim_{x\to 3} \frac{x+3}{2x+1} = \frac{3+3}{2\cdot 3+1} = \frac{6}{7}$

(d) $\lim_{x\to -1^-} \frac{8-8x}{1-x^2} = -\infty$

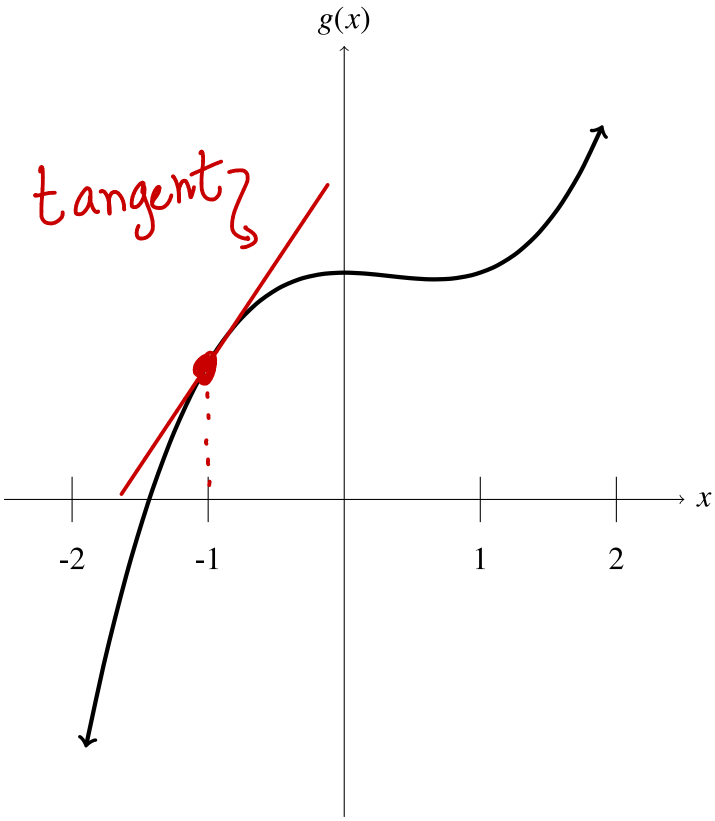
form $\frac{16}{0}$

as $x \to -1^-$ (so x-values like -1.001)

$1-x^2 \to 0^-$ and $8-8x \to 16$

So the quotient is $-\infty$.

5. (12 points) The function $g(x) = x^2(x-1) + 4$ is graphed below.



$g(x) = x^3 - x^2 + 4$

$g'(x) = 3x^2 - 2x$

$g'(-1) = 3(-1)^2 - 2(-1)$

$= 3 + 2 = 5$

$g(-1) = (-1)^3 - (-1)^2 + 4$

$= -1 - 1 + 4$

$= 2$

(a) Sketch and label the **tangent** line to the graph of $g(x)$ when $x = -1$.

(b) Write an equation of the tangent line to $g(x)$ when $x = -1$.

point $(-1, 2)$

slope $m = 5$

line: $y - 2 = 5(x+1)$

$y = 2 + 5(x+1)$

(c) Find all x-values where the graph of $g(x)$ has a horizontal tangent or explain why none exist.

Find x-values where $g'(x) = 3x^2 - 2x = 0$

So $x(3x-2) = 0$.

So $x = 0$ or $x = 2/3$

6. (15 points) Find the derivative of each function below. You do not need to simplify your answer. Be careful to appropriately parenthesize your answer.

   (a) $f(x) = 2x^5 + \frac{2}{x^5} + 2\sqrt{5}$ $= 2x^5 + 2x^{-5} + 2\sqrt{5}$

   $f'(x) = 10x^4 - 10x^{-6}$

   (b) $H(\theta) = \quad \theta^{2/3}\sin(\theta)$

   $H'(\theta) = \quad \frac{2}{3}\theta^{-1/3} \cdot \sin(\theta) + \theta^{2/3}\cos(\theta)$

   (c) $j(x) = \frac{x}{\pi + \cos(x)}$

   $j'(x) = \frac{(\pi + \cos(x))(1) - x(-\sin(x))}{(\pi + \cos(x))^2} = \frac{\pi + \cos(x) + x\sin(x)}{(\pi + \cos(x))^2}$

7. (6 points) The function $V(T)$ models the volume of a fixed mass of a given gas with respect to temperature assuming pressure remains constant. Assume V is measured in milliliters (or mL) and T is measured in kelvins (or K).

   (a) Interpret in the context of the problem the meaning of $V(100) = 60$. Include units.

   At a temperature of 100 Kelvins, the volume of the gas is 60 mL.

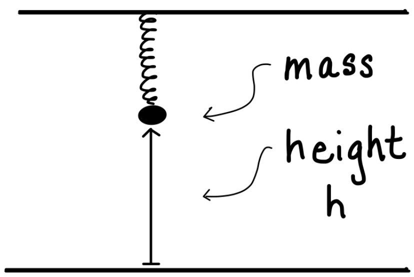
   (b) Interpret in the context of the problem the meaning of $V'(100) = 0.4$. Include units.

   At a temperature of 100 kelvins, the volume of gas is increasing at a rate of 0.4 mL per kelvin.

   (c) Using the facts that $V(100) = 60$ and $V'(100) = 0.4$, estimate $V(110)$. Include units.

   $V(110) = V(100) + 10 \cdot V'(100) = 60 + 10(0.4) = 64\,K.$

8. (17 points) A spring is hanging from the ceiling with a mass attached to it. The mass is oscillating vertically with simple harmonic motion. The function $h(t) = 6 - \cos(t)$ models the height of the spring above the floor starting at time $t = 0$ where h is measured in feet and t is measured in seconds. **For all problems below, include appropriate units.**



$$h(t) = 6 - \cos(t)$$
$$h(\pi) = 6 - \cos(\pi) = 7$$
$$h(0) = 6 - \cos(0) = 5$$

(a) Find the average velocity of the mass in the time interval $[0, \pi]$ seconds.

$$\text{avg vel} = \frac{\Delta h}{\Delta t} = \frac{h(\pi) - h(0)}{\pi - 0} = \frac{7-5}{\pi} = \frac{2}{\pi} \text{ ft/s}$$

(b) Find the equations for the velocity and acceleration of the mass.

$$h'(t) = \sin(t) \quad \text{ft/s}$$
$$h''(t) = \cos(t) \quad \text{ft/s}^2$$

(c) Find the instantaneous velocity of the mass at $\pi/4$ seconds.

$$h'\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \sqrt{2}/2 \quad \text{ft/s}$$

(d) At $t = \pi/4$, is the mass going up or going down?

going up

(e) At what times does the mass ~~reach its lowest position?~~ change direction?

Check for $v'(t) = 0$.

Solve $\sin(t) = 0$

So $t = 0, \pi, 2\pi, 3\pi, \ldots$

Extra Credit: (5 points) Use the Intermediate Value Theorem to demonstrate that the function $f(x) = x^3 - 3x - 19$ has a root (or zero) in the interval $[1, 5]$. A complete answer requires some calculations **and** complete sentences justifying your conclusion.

Since $f(x)$ is a polynomial, it is continuous and the Intermediate Value Theorem applies.

Observe that $f(1) = 1^3 - 3(1) - 19 < 0$ and

$$f(5) = 5^3 - 3\cdot 5 - 19 > 0.$$

Since $f(1) < 0$ and $f(5) > 0$, the Intermediate Value Theorem implies that $f(x) = 0$ for some x-value between 1 and 5.