Math F251 Midterm 2 Spring 2023

Name: Solutions

Rules:

You have 90 minutes to complete the exam.

Partial credit will be awarded, but you must show your work.

You may have a single handwritten 3×5 notecard.

Calculators are not allowed.

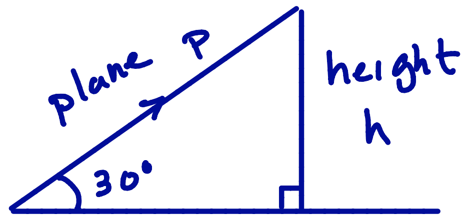
Place a box around your $\boxed{\text{FINAL ANSWER}}$ to each question where appropriate.

Turn off anything that might go beep during the exam.

Good luck!

Problem	Possible	Score
1	10	
2	10	
3	12	
4	14	
5	16	
6	10	
7	8	
8	8	
9	12	
Extra Credit	5	
Total	100	

1. (10 points) If a airplane is flying at a constant speed of 100 miles per hour and is climbing at a an angle of 30 degrees, at what rate is its altitude changing? Your final answer should include units.



Setup

$\sin(30°) = \frac{h}{p}$

want $\frac{dh}{dt}$

Know $\frac{dp}{dt} = 100\text{mph}$

Recall $\sin(30°) = \frac{1}{2}$.

Use $\frac{1}{2}p = h$.

So $\frac{1}{2}\frac{dp}{dt} = \frac{dh}{dt}$

So $\frac{1}{2}100 = \frac{dh}{dt}$

Answer: The height is increasing at a rate of 50mph.

2. (10 points)

   (a) Find the linear approximation, $L(x)$, to $f(x) = \sqrt{x}$ at $a = 25$.

   $f(25) = \sqrt{25} = 5$

   $f'(x) = \frac{1}{2} x^{-1/2}$

   $f'(25) = \frac{1}{2} \cdot \frac{1}{\sqrt{25}} = \frac{1}{10}$

   $y - 5 = \frac{1}{10}(x-25)$

   $y = 5 + \frac{1}{10}(x-25)$

   $\boxed{L(x) = 5 + \frac{1}{10}(x-25)}$

   (b) Use your answer in part a to estimate $\sqrt{23}$. Your answer should be in the form of a simplified fraction or a decimal.

   $\sqrt{23} \approx L(23) = 5 + \frac{1}{10}(23-25) = 5 - \frac{2}{10} = \frac{48}{10} = 4.8$

3. (12 points) Evaluate the indefinite integrals below.

   (a) $\int \left(8x^{2/3} + \sec(x)\tan(x)\right)\,dx = \frac{8x^{5/3}}{\frac{5}{3}} + \sec(x) + C = \frac{24}{5} x^{5/3} + \sec(x) + C$

   (b) $\int (e^x + 2)\,dx = e^x + 2x + C$

   (c) $\int \frac{1+x^3}{x^2}\,dx = \int (x^{-2} + x)\,dx = -x^{-1} + \frac{1}{2}x^2 + C$

4. (14 points) Let $f(x) = \ln(x^2 + 2)$. It is a fact that $f'(x) = \dfrac{2x}{x^2+2}$ and $f''(x) = \dfrac{-2(x^2-2)}{(x^2+2)^2}$.

(a) Determine intervals where $f(x)$ is increasing or decreasing.

$f'(x) = 0$ at $x = 0$

$- - -\ 0\ + + +$ ← sign of f'

$-1 \quad 0 \quad 1$

$f'(-1) = \frac{-}{+}$

$f'(1) = \frac{+}{+}$

Answer:

f is increasing on $(0, \infty)$.

f is decreasing on $(-\infty, 0)$

(b) Determine the x-values of any local maxima or minima or state that none exist. and state the maximum or minimum values.

Answer: $f(x)$ has a minimum of $\ln(2)$ at $x = 0$.

$f(x)$ has no maximum

(c) Determine intervals where $f(x)$ is concave up or concave down.

$f'' = 0$ when $x = \pm\sqrt{2}$

$- - - -\ 0\ + + +\ 0\ - - -$ ← sign of f''

$-10 \quad -\sqrt{2} \quad 0 \quad \sqrt{2} \quad 10$

$f''(-10) = \frac{(-)(+)}{+} = -$

$f''(0) = \frac{(-)(-)}{+} = +$

$f''(10) = \frac{(-)(+)}{+} = -$

Answer: f is cc up on $(-\sqrt{2}, \sqrt{2})$, cc down on $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$

(d) ~~Determine the x-values~~ Identify of all inflection points or state that none exist.

$(-\sqrt{2}, f(-\sqrt{2})) = (-\sqrt{2}, \ln(4))$

$(\sqrt{2}, f(\sqrt{2})) = (\sqrt{2}, \ln(4))$

5. (16 points) Evaluate the limits below. You must show your work. Indicate an application of L'Hôpital's Rule by putting an H above equal sign.

(a) $\displaystyle\underbrace{\lim_{x\to 0}\frac{e^x-\cos(x)}{4\tan(x)}}_{\text{form } \frac{0}{0}} \overset{H}{=} \lim_{x\to 0}\frac{e^x+\sin(x)}{4\sec^2(x)} = \frac{e^0+\sin(0)}{4\sec^2(0)}$

$$= \frac{1+0}{4\cdot 1} = \frac{1}{4}$$

(b) $\displaystyle\underbrace{\lim_{x\to\infty}\frac{\ln(x)}{x^{1/2}}}_{\text{form } \frac{\infty}{\infty}} \overset{H}{=} \lim_{x\to\infty}\frac{\frac{1}{x^{-1}}}{\frac{1}{2}x^{-1/2}} = \lim_{x\to\infty}\frac{2\sqrt{x}}{x} = \lim_{x\to\infty}\frac{2}{\sqrt{x}} = 0$

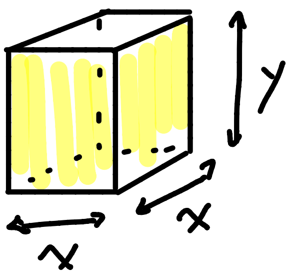
(c) $\displaystyle\lim_{x\to 0^+}\left(1+x\right)^{\frac{1}{2x}} = \boxed{e^{1/2}}$

change

$$\lim_{x\to 0^+}\frac{1}{2x}\ln(1+x) = \underbrace{\lim_{x\to 0^+}\frac{\ln(1+x)}{2x}}_{\text{form } 0/0} \overset{H}{=} \lim_{x\to 0^+}\frac{\frac{1}{1+x}}{2} = \frac{1}{2}$$

Justify your answer

6. (10 points) An open-topped box with a square base has a fixed surface area of 1200 in^2. Determine the dimensions of the box with maximum volume.



Goal: Maximize volume $V = x^2 y$

Constraint:

$$\text{surface area} = S = x^2 + 4xy = 1200$$

$$y = \frac{1200 - x^2}{4x} = 300x^{-1} - \frac{1}{4}x$$

Function: $V(x) = x^2\left(300x^{-1} - \frac{1}{4}x\right) = 300x - \frac{1}{4}x^3$

domain: $(0, \infty)$

Find crit #'s: $V'(x) = 300 - \frac{3}{4}x^2 = 0$

$$300 = \frac{3}{4}x^2$$

$$x^2 = \frac{4 \cdot 300}{3} = 400$$

$x = \pm 20$ (only the positive value in domain)

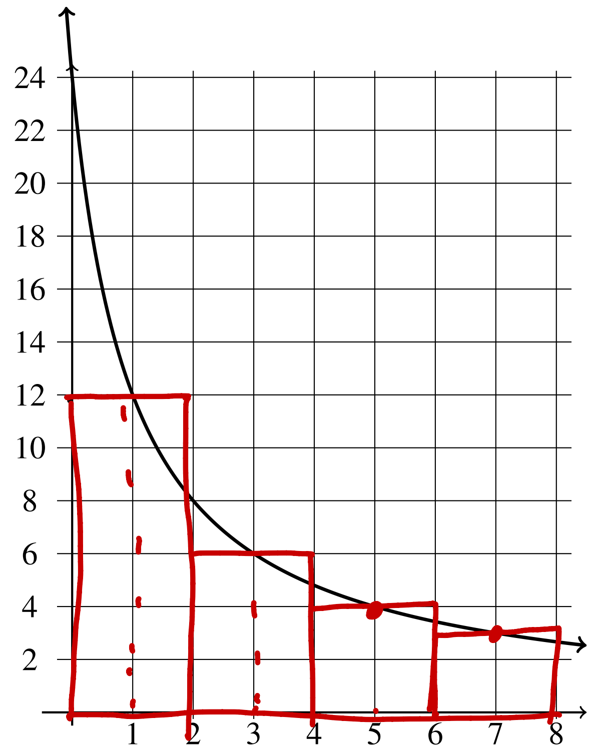
Justify: $V''(x) = -\frac{6}{4}x$, $V''(20) < 0$. So V is ccdown at $x = 20$ is the location of a maximum.

Find y. $y = \frac{1200 - 400}{80} = \frac{800}{80} = 10$

Answer: The dimensions that maximize volume are 20 in × 20 in × 10 in.

7. (8 points) The function $f(x) = \frac{24}{x+1}$ is graphed below. We want to estimate the area under the curve $f(x)$ on the interval $[0, 8]$ using M_4. (That is, we want to use 4 approximating rectangles with midpoints determining height.)

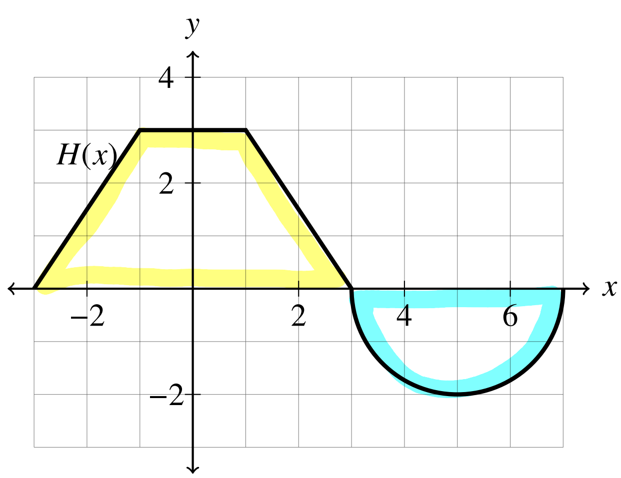
   ✓ (a) Sketch the four approximating rectangles on the graph.



   (b) Do a calculation to estimate the area under the curve using M_4 (that is, use 4 approximating rectangles and midpoints) and simplify your answer.

$$A \approx M_4$$
$$= 2(f(1)+f(3)+f(5)+f(7))$$
$$= 2(12+6+4+3)$$
$$= 2(25) = \underline{\underline{50}}$$

8. (8 points) Evaluate the definite integrals below using the graph of $H(x)$ and properties of definite integrals. On the interval $[3, 7]$, the graph of H is a semi-circle. Show your work.



   (a) $\int_{-3}^{7} f(x)\,dx = \left(\text{area above x-axis}\right) - \left(\text{area below x-axis}\right)$

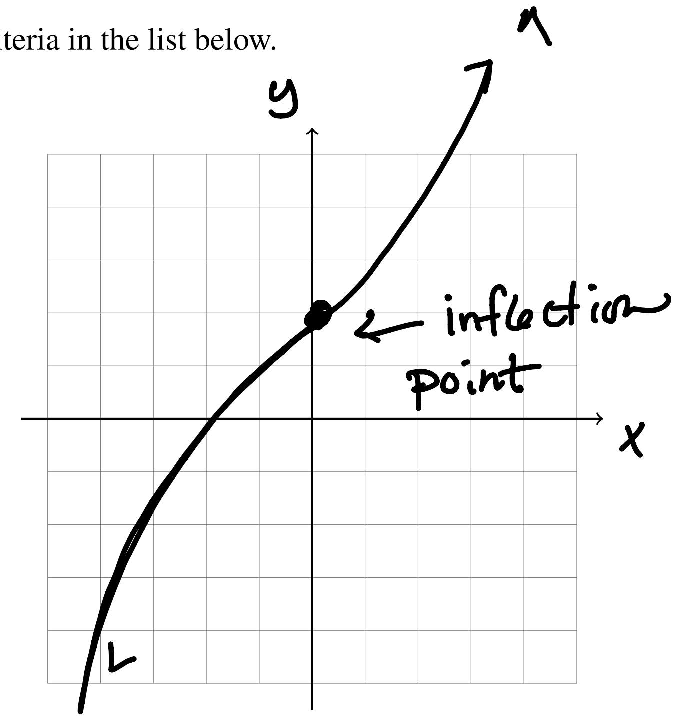
$$= 12 - \frac{1}{2}\pi \cdot 2^2 = 12 - 2\pi$$

   (b) $\int_0^3 (4f(x) + 6)\,dx = 4\int_0^3 f(x) + 6\int_0^3 1\,dx$

$$= 4(6) + 6(3) = 24 + 18 = 42$$

$$\begin{array}{r} \overset{1}{2}4 \\ 18 \\ \hline 42 \end{array}$$

9. (12 points) Sketch a graph that satisfies all the criteria in the list below.



- Domain $(-\infty, \infty)$
- $f(0) = 2$ point $(0,2)$
- $f'(x) > 0$ f is ↑ ~~↗~~
- $\underbrace{f''(x) < 0 \text{ on } (-\infty, 1)}_{\text{cc down}}$, to $\underbrace{f''(x) > 0 \text{ on } (1, \infty)}_{\text{ccup}}$

Extra Credit (5 points) Identify all vertical and horizontal asymptotes of the function $f(x) = \frac{4e^x+1}{7e^x-1}$. Justify your answer using limits.

horizontal asymptotes: $y = \frac{4}{7}$ and $y = -1$

Justification: $\lim_{x\to\infty} \frac{4e^x+1}{7e^x-1} = \frac{4}{7}$; $\lim_{x\to-\infty} \frac{4e^x+1}{7e^x-1} = \frac{1}{-1} = -1$

vertical asymptotes: $x = \ln(1/7)$ [Obtained by solving: $7e^x - 1 = 0$ or $e^x = 1/7$]

Justification: $\lim_{x\to \ln(1/7)^+} \frac{4e^x+1}{7e^x-1} = +\infty$

since $4e^x + 1 > 0$ forall x and as $x \to \ln(\frac{1}{7})^+$, $7e^x - 1 \to 0^+$.