Spring 2024 **Math F251X**

Calculus I: Final Exam

Name: Solutions

Section: ☐ **9:15 (Mohamed Nouh)**
☐ **11:45 (James Gossell)**
☐ **Online (Leah Berman)**

Rules:

- Partial credit will be awarded, but you must **show your work**.
- You may have a single handwritten 3″ × 5″ notecard, both sides.
- Calculators are **not allowed**.
- Place a box around your FINAL ANSWER to each question where appropriate.
- Turn off anything that might go beep during the exam.
- You have two hours to complete the exam.

Problem	Possible	Score
1	12	
2	10	
3	10	
4	12	
5	11	
6	10	
7	8	
8	12	
9	11	
10	4	
Extra Credit	(5)	
Total	100	

v1

1. (12 points)

Compute the following **integrals**. Give the most general answer, and show your work. Clearly indicate any substitutions you use in such a way that someone else can follow your work.

a. $\int \sqrt[4]{x^3} + \sqrt{2} - \sin(x)\,dx$

$= \int x^{3/4} + \sqrt{2} - \sin(x)\,dx$

$= \dfrac{x^{7/4}}{7/4} + x\sqrt{2} - (-\cos(x)) + C$

$= \boxed{\frac{4}{7}x^{7/4} + x\sqrt{2} + \cos(x) + C}$

b. $\int \dfrac{2x\ln(x^2+1)}{x^2+1}\,dx$

① $u = x^2+1$

$\dfrac{du}{dx} = 2x \Rightarrow \dfrac{du}{2x} = dx$

it would have been earier to just sub $u = \ln(x^2+1)$ at the start!

$= \int \dfrac{\cancel{2x}\ln(u)}{u} \cdot \dfrac{du}{\cancel{2x}}$

$= \int \dfrac{\ln(u)}{u}\,du$

② $v = \ln(u)$

$dv = \frac{1}{u}\,du$

$= \int v\,dv = \dfrac{v^2}{2} + C = \dfrac{(\ln(u))^2}{2} + C = \boxed{\frac{1}{2}(\ln(x^2+1))^2 + C}$

c. $\int \dfrac{x^2}{\sqrt{1+x^3}}\,dx$

$u = 1+x^3 \Rightarrow \dfrac{du}{dx} = 3x^2 \Rightarrow \dfrac{du}{3x^2} = dx$

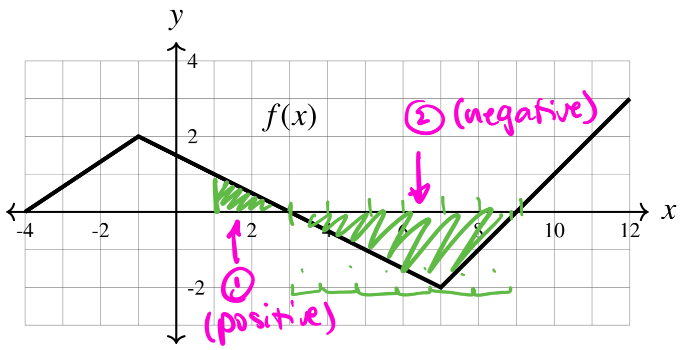
$= \int \dfrac{\cancel{x^2}}{\sqrt{u}} \cdot \dfrac{du}{3\cancel{x^2}}$

$= \frac{1}{3}\int u^{-1/2}\,du$

$= \frac{1}{3}\dfrac{u^{1/2}}{1/2} + C = \frac{2}{3}u^{1/2} + C = \boxed{\frac{2}{3}(1+x^3)^{1/2} + C} = \frac{2}{3}\sqrt{1+x^3} + C$

2. (10 points)

Consider the graph of the function $f(x)$ shown below:



a. Compute $\int_1^9 f(x)\,dx$. Show some work or say something about what you computed.

= area of shaded part

= ① + ②

$= \frac{1}{2}(1 \cdot 2) - \frac{1}{2}(6 \cdot 2)$

$= 1 - 6$

$= -5$

b. Let $F(t) = \int_{-4}^{t} f(x)\,dx$. **On the interval** $[-4, 12]$, where is $F(t)$ increasing? Where is $F(t)$ decreasing? Write your answers in interval notation.

- $F(t)$ is **increasing** on the interval $[-4, 3] \cup [9, 12]$
- $F(t)$ is **decreasing** on the interval $[3, 9]$

c. Determine $f'(1) =$ $-1/2$ = slope of TL at $x = 1$

d. Determine

(i) $\lim_{x \to -1^-} f'(x) =$ $2/3$

(ii) $\lim_{x \to -1^+} f'(x) =$ $-1/2$

(iii) $\lim_{x \to -1} f'(x) =$ DNE because the LH & RH limits disagree

3. (10 points)

A portion of the implicitly defined curve

$$y^2 = x^3 - 3x + 3$$

is shown in the graph.

a. Use implicit differentiation to find the **slope** of the tangent line to the curve at the point $(-2, 1)$ (shown as the black dot). Please give an exact answer.

$\frac{d}{dx}(y^2) = \frac{d}{dx}(x^3 - 3x + 3) \Rightarrow$

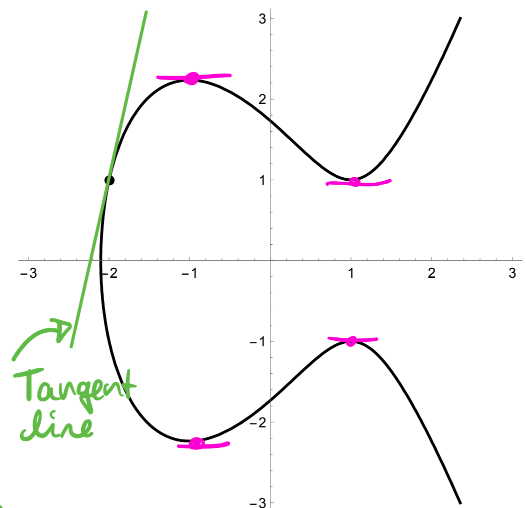
$2y\frac{dy}{dx} = 3x^2 - 3$

So at $y = 1$, $x = -2$:

$2(1)\frac{dy}{dx} = 3(-2)^2 - 3 \Rightarrow$

$2\frac{dy}{dx} = 9 \Rightarrow \frac{dy}{dx} = \frac{9}{2}$

(which is plausible given the picture!)



b. **Write the equation** of the tangent line to the curve at the point $(-2, 1)$, which is shown with a black dot on the curve. **Clearly draw and label** the tangent line on the graph.

$y = \frac{9}{2}(x + 2) + 1$

c. Find the coordinates (as ordered pairs) of all points on the curve where the tangent line is horizontal.

We need $\frac{dy}{dx} = 0$. From above, $2y\frac{dy}{dx} = 3x^2 - 3 \Rightarrow$ $0 = 3x^2 - 3 \Rightarrow 3x^2 = 3 \Rightarrow x^2 = 1 \Rightarrow x = 1$ or $x = -1$. In the original equation, $y^2 = (1)^3 - 3(1) + 3 = 1 - 3 + 3 = 1 \Rightarrow y = \pm 1$, or $y^2 = (-1)^3 - 3(-1) + 3 = -1 + 3 + 3 = 5 \Rightarrow y = \pm\sqrt{5}$.

Coordinates of points: $(1,1), (1,-1), (-1,\sqrt{5}), (-1,-\sqrt{5})$

(these are plausible from the graph!)

4. (12 points)

A surveillance drone rises from the ground. Its upward velocity is can be modelled by the function

$$v(t) = 1 - e^{-t}$$

meters per second at the instant that it is t seconds into its flight.

a. Compute $v(0)$. Write a complete sentence explaining the meaning of $v(0)$ in the context of the problem. Include units in your answer.

$v(0) = 1 - e^{-0} = 1 - 1 = 0$. At time $t = 0$, the drone is travelling at 0 m/s (it is at rest).

b. Compute $\int_0^5 v(t)dt$ and write a sentence to interpret its meaning in the context of the problem. Include units in your answer.

$u = -t \quad du = -dt$

$\int_0^5 1 - e^{-t}\, dt = \int_0^5 1\, dt - \int_0^5 e^{-t}\, dt = 5 + \int_{-0}^{-5} e^u\, du$

$= 5 + e^u \Big|_{-0}^{-5} = 5 + e^{-5} - e^0 = 4 + 1/e^5$

Between 0 and 5 seconds, the drone travelled $4 + \frac{1}{e^5}$ meters.

c. Find $v'(t)$. Compute $v'(0)$ and write a sentence to interpret its meaning in the context of the problem. Include units in your answer.

$v'(t) = 0 - (e^{-t})(-1) = e^{-t}$, so $v'(0) = 1$

At time $t=0$, the drone is accelerating at a rate of $1\ m/s^2$.

5. (11 points)

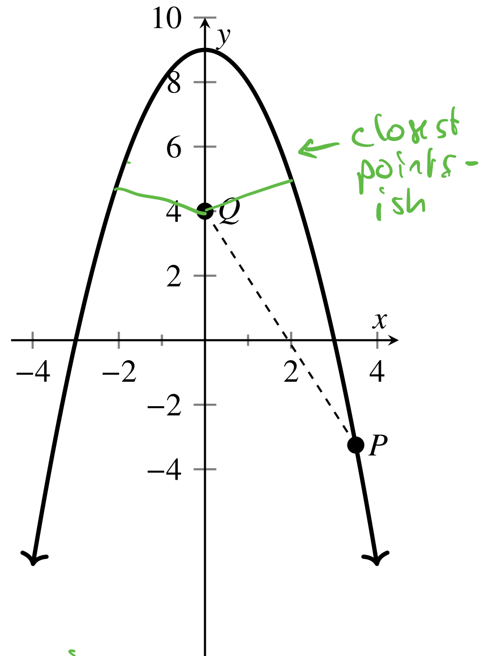
We want to answer the following question: which points on the graph of $y = 9 - x^2$ are **closest** to the point $(0, 4)$?

It is a mathematical fact that to minimize the distance between two points, it is sufficient to minimize the **square** of the distance. The function

$$D(x) = (x - 0)^2 + ((9 - x^2) - 4)^2 = x^4 - 9x^2 + 25$$

gives the square of the distance between the point $Q = (0, 4)$ and an arbitrary point $P = (x, y) = (x, 9 - x^2)$ on the graph.

Determine the x-value(s) which **minimize** $D(x)$ in order to find the point(s) on the graph that are closest to the point (0,4). Use Calculus to justify your answer.



$D'(x) = 4x^3 - 18x = 2x(2x^2 - 9)$

$D'(x) = 0 \Rightarrow 2x(2x^2-9) = 0 \Rightarrow$

$x = 0$ or $x^2 = 9/2 \Rightarrow x = \frac{3}{\sqrt{2}}$ or $x = \frac{-3}{\sqrt{2}}$.

Are there local minima? Use 2^{nd} deriv test:

$D''(x) = 12x^2 - 18$

$D''\left(\frac{3}{\sqrt{2}}\right) = 12\left(\frac{9}{2}\right) - 18 = 6 \cdot 9 - 18 > 0$

$\Rightarrow \cup \Rightarrow$ min.

$D''\left(-\frac{3}{\sqrt{2}}\right) = 12\left(\frac{9}{2}\right) - 18 > 0$ as well $\Rightarrow \cup \Rightarrow$ min.

Plug these into the original equation $y = 9 - x^2$:

When $x = \frac{3}{\sqrt{2}}$, $y = 9 - \left(\frac{3}{\sqrt{2}}\right)^2 = \frac{18}{2} - \frac{9}{2} = \frac{9}{2}$

When $x = \frac{-3}{\sqrt{2}}$, $y = 9 - \left(\frac{-3}{\sqrt{2}}\right)^2 = \frac{18}{2} - \frac{9}{2} = 9/2$

The closest point(s) is/are : $\left(3/\sqrt{2}, 9/2\right)$ and $\left(\frac{-3}{\sqrt{2}}, 9/2\right)$

(your answer should be (an) ordered pair(s)!)

6. (10 points)

A giant spherical snowball is melting. Its volume is decreasing at a rate of 1 cubic meter per hour. **How fast** is the radius of the snowball decreasing when its radius is 2 meters? (The volume V of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$.)

Write your answer in a complete sentence using units.

$V = \frac{4\pi}{3} \cdot r^3$

$\frac{dV}{dt} = \frac{4\pi}{3} \cdot 3r^2 \frac{dr}{dt}$

Know $\frac{dV}{dt} = -1\, m^3/hour$ (it's decreasing!)

Want $\frac{dr}{dt}$ when $r = 2m$

So $-1 = \frac{4\pi}{\cancel{3}} \cdot \cancel{3}(2)^2 \frac{dr}{dt} \Rightarrow$

$-1 = 16\pi \frac{dr}{dt} \Rightarrow$

$\frac{dr}{dt} = \frac{-1}{16\pi}$

When the radius is 2 meters, the radius is decreasing at a rate of $1/16\pi$ m/s (that is, the rate is changing at a rate of $-1/16\pi$ m/s)

7. (8 points)

Compute the following **limits**. Show your work clearly. Make sure you use **limit notation** where required; an answer that does not use proper notation will not receive full credit. Use = to show things are equal. If you use L'Hôpital's rule, write $\overset{H}{=}$ or $\overset{L'H}{=}$ to indicate where you are applying it.

a. $\lim_{x\to\infty} \frac{\ln(2x^4+3)}{x^4}$ type $\frac{\infty}{\infty}$

$\overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{2x^4+3}(8x^3)}{4x^3} = \lim_{x\to\infty} \frac{\cancel{8x^3}^{\,2}}{2x^4+3}\cdot\frac{1}{\cancel{4x^3}} = \lim_{x\to\infty} \frac{2}{2x^4+3} = \boxed{0}$

b. $\lim_{x\to 1} \frac{\sqrt{2-x}-x}{x-1}$ type $\frac{1-1}{1-1} = 0/0$ $= \lim_{x\to 1} \frac{(2-x)^{1/2}-x}{x-1}$

$\overset{L'H}{=} \lim_{x\to 1} \frac{1/2\,(2-x)^{-1/2}(-1)-1}{1} = \left(\frac{1}{2}(2-1)^{-1/2}(-1)\right)-1 = \frac{1}{2}(1)(-1)-1 = -1-1/2 = \boxed{-3/2}$

8. (12 points)

Compute the following **derivatives**. Show your work. You do NOT need to simplify your answer. Your answer should start $f'(x)$, $\frac{df}{dx}$ etc.

a. $f(\theta) = \ln(\sec\theta + \cot\theta)$

$f'(\theta) = \left(\frac{1}{\sec\theta+\cot\theta}\right)(\sec\theta\tan\theta - \csc^2\theta)$

b. $g(x) = \frac{\sqrt{3}}{4} + \frac{\sqrt{x}}{5} - \frac{5}{\sqrt{x}}$ $= \frac{\sqrt{3}}{4} + \frac{1}{5}x^{1/2} - 5x^{-1/2}$

$g'(x) = 0 + \frac{1}{5}\left(\frac{1}{2}x^{-1/2}\right) - 5\left(-1/2\,x^{-3/2}\right)$

c. $h(x) = \frac{e^{3x}}{\sin(x)}$

$h'(x) = \frac{\sin(x)\,(e^{3x}(3)) - e^{3x}\cos(x)}{(\sin(x))^2}$

9. (11 points)

Sketch a graph of a function $h(x)$ that satisfies all of the following properties.

After drawing the graph:

- **Label** on the graph the following things, if they exist, by drawing a point on the graph and labeling: any local maximums by writing LOCAL MAX, local minimums by writing LOCAL MIN, inflection points by writing IP
- **Draw** any horizontal and vertical asymptotes with dashed lines and **label** them with their equation.
- **Mark** any important x-values and y-values (with numbers) on the x- and y-axes.

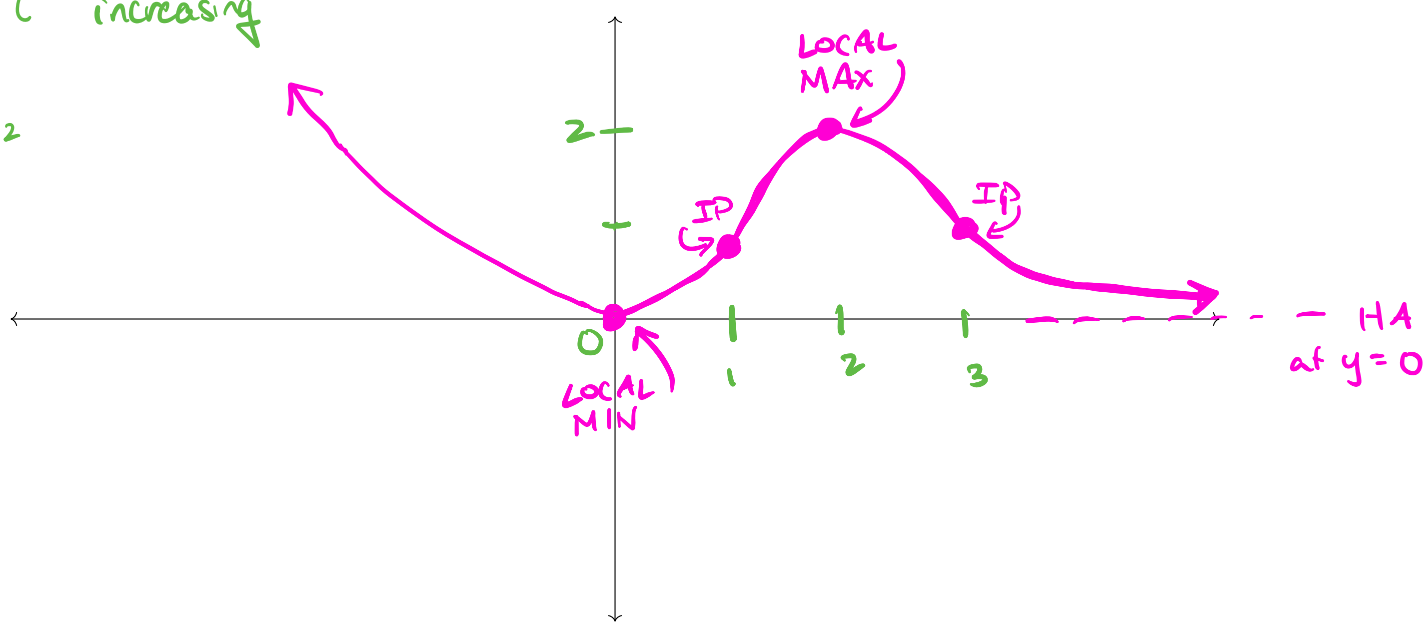
Properties:

↘ ↗ at $x=0$

- The domain of $h(x)$ is $(-\infty, \infty)$
- $h(0) = 0$ and $h(2) = 2$
- $h'(x) < 0$ on the interval $(-\infty, 0) \cup (2, \infty)$
  decreasing
- $h'(x) > 0$ on the interval $(0, 2)$
  increasing
- $h''(x) < 0$ on the interval $(1, 3)$
  CD
- $h''(x) > 0$ on the interval $(-\infty, 1) \cup (3, \infty)$
  CU
- $\lim_{x\to-\infty} h(x) = \infty$ ↖ as y ←
- $\lim_{x\to\infty} h(x) = 0$ HA at $y=0$ →

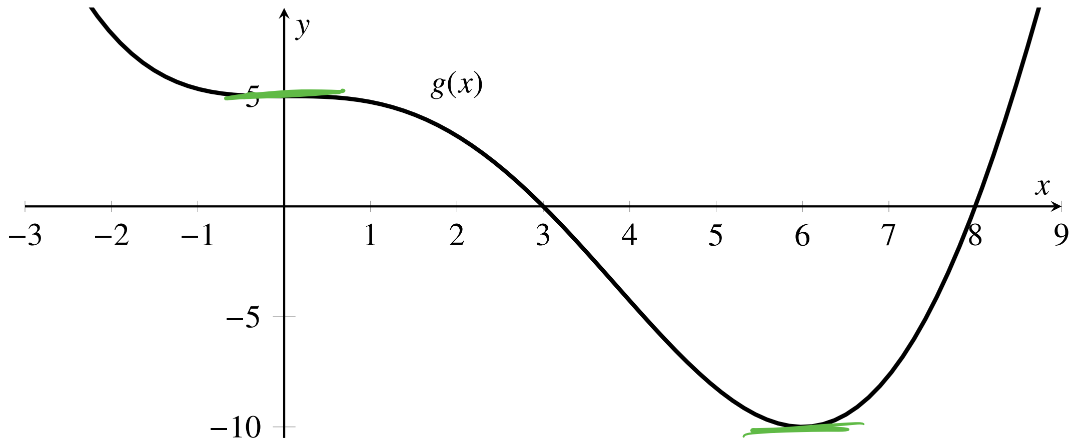
IP at $x=1$, $x=3$

↗ ↘ at $x=2$

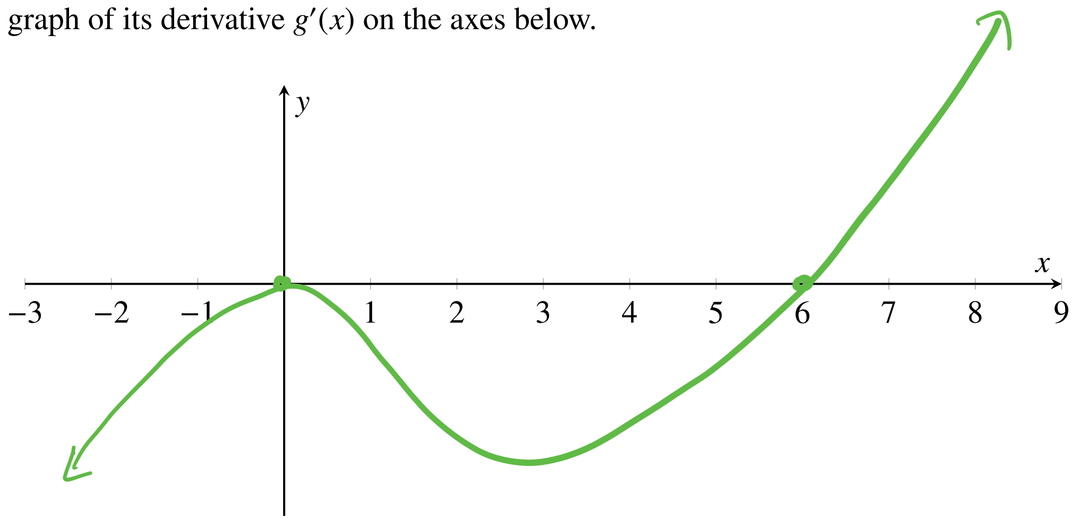


10. (4 points)

The graph of a function $g(x)$ is shown below.

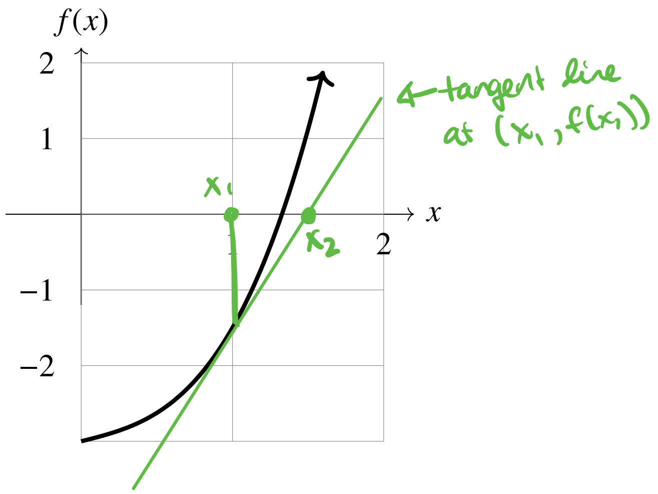


Sketch the graph of its derivative $g'(x)$ on the axes below.



Extra Credit (5 points) A portion of the graph of the function $f(x) = -3 + x/2 + x^3$ is shown below.

$= -3 + \frac{1}{2}x + x^3$



a. Suppose Newton's method is used to find an approximate solution to $f(x) = 0$ from an initial guess of $x_1 = 1$. **Sketch** on the graph how the next approximation x_2 will be found, **labeling** its location on the x-axis.

re-deriving Newton's formula:

$y = f'(x_1)(x - x_1) + f(x_1) = 0 \Rightarrow$

$f'(x_1)(x - x_1) = -f(x_1) \Rightarrow$

$x - x_1 = \frac{-f(x_1)}{f'(x_1)} \Rightarrow x = x_1 - \frac{f(x_1)}{f'(x_1)}$

b. If your starting guess is $x_1 = 1$, **compute** x_2.

$f(1) = -3 + \frac{1}{2} + 1^3 = -2 + 1/2 = -3/2$

$f'(x) = \frac{1}{2} + 3x^2$

$f'(1) = \frac{1}{2} + 3 = \frac{7}{2}$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \Rightarrow x_2 = 1 - \frac{-3/2}{7/2} = 1 + 3/7 = \frac{10}{7}$ (plausible)