Spring 2024 Math F251X

Calculus I: Final Exam

Name: ______________________

Section: ☐ 9:15 (Mohamed Nouh)
☐ 11:45 (James Gossell)
☐ Online (Leah Berman)

Rules:

- Partial credit will be awarded, but you must **show your work**.
- You may have a single handwritten $3'' \times 5''$ notecard, both sides.
- Calculators are **not allowed**.
- Place a box around your FINAL ANSWER to each question where appropriate.
- Turn off anything that might go beep during the exam.
- You have two hours to complete the exam.

Problem	Possible	Score
1	12	
2	10	
3	10	
4	12	
5	11	
6	10	
7	8	
8	12	
9	11	
10	4	
Extra Credit	(5)	
Total	100	

1. (12 points)

Compute the following **integrals**. Give the most general answer, and show your work. Clearly indicate any substitutions you use in such a way that someone else can follow your work.

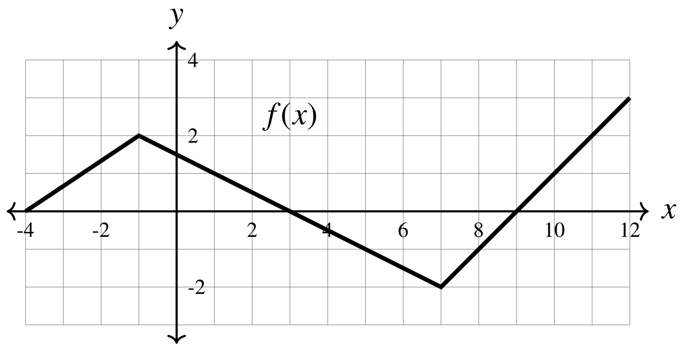
a. $\int \sqrt[4]{x^3} + \sqrt{2} - \sin(x)\, dx$

b. $\int \frac{2x \ln(x^2+1)}{x^2+1}\, dx$

c. $\int \frac{x^2}{\sqrt{1+x^3}}\, dx$

2. (10 points)

Consider the graph of the function $f(x)$ shown below:



a. Compute $\int_1^9 f(x)\,dx$. Show some work or say something about what you computed.

b. Let $F(t) = \int_{-4}^{t} f(x)\,dx$. **On the interval** $[-4, 12]$, where is $F(t)$ increasing? Where is $F(t)$ decreasing? Write your answers in interval notation.

- $F(t)$ is **increasing** on the interval ______________________
- $F(t)$ is **decreasing** on the interval ______________________

c. Determine $f'(1) =$ __________

d. Determine

(i) $\lim_{x\to -1^-} f'(x) =$ __________

(ii) $\lim_{x\to -1^+} f'(x) =$ __________

(iii) $\lim_{x\to -1} f'(x) =$ __________

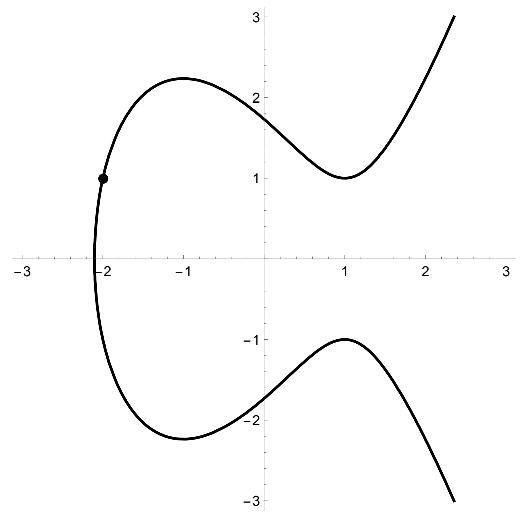
3. (10 points)

A portion of the implicitly defined curve

$$y^2 = x^3 - 3x + 3$$

is shown in the graph.

a. Use implicit differentiation to find the **slope** of the tangent line to the curve at the point $(-2, 1)$ (shown as the black dot). Please give an exact answer.



b. **Write the equation** of the tangent line to the curve at the point $(-2, 1)$, which is shown with a black dot on the curve. **Clearly draw and label** the tangent line on the graph.

c. Find the coordinates (as ordered pairs) of all points on the curve where the tangent line is horizontal.

Coordinates of points: ______________________________

4. (12 points)

A surveillance drone rises from the ground. Its upward velocity is can be modelled by the function

$$v(t) = 1 - e^{-t}$$

meters per second at the instant that it is t seconds into its flight.

a. Compute $v(0)$. Write a complete sentence explaining the meaning of $v(0)$ in the context of the problem. Include units in your answer.

b. Compute $\int_0^5 v(t)dt$ and write a sentence to interpret its meaning in the context of the problem. Include units in your answer.

c. Find $v'(t)$. Compute $v'(0)$ and write a sentence to interpret its meaning in the context of the problem. Include units in your answer.

5. (11 points)

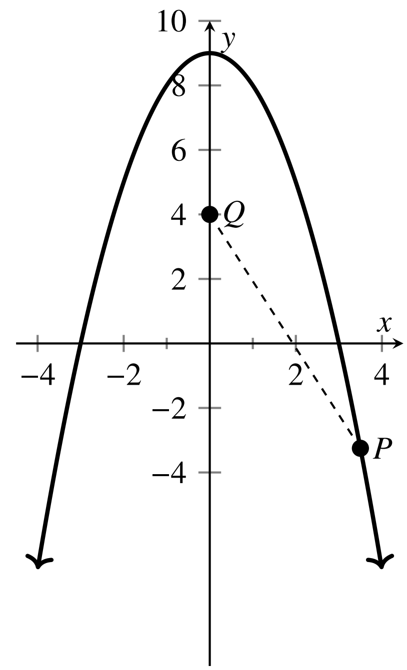
We want to answer the following question: which points on the graph of $y = 9 - x^2$ are **closest** to the point $(0, 4)$?

It is a mathematical fact that to minimize the distance between two points, it is sufficient to minimize the **square** of the distance. The function

$$D(x) = (x - 0)^2 + ((9 - x^2) - 4)^2 = x^4 - 9x^2 + 25$$

gives the square of the distance between the point $Q = (0, 4)$ and an arbitrary point $P = (x, y) = (x, 9 - x^2)$ on the graph.

Determine the x-value(s) which **minimize** $D(x)$ in order to find the point(s) on the graph that are closest to the point (0,4). Use Calculus to justify your answer.



The closest point(s) is/are : ______________________

(your answer should be (an) ordered pair(s)!)

6. (10 points)

A giant spherical snowball is melting. Its volume is decreasing at a rate of 1 cubic meter per hour. **How fast** is the radius of the snowball decreasing when its radius is 2 meters? (The volume V of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$.)

Write your answer in a complete sentence using units.

7. (8 points)

Compute the following **limits**. Show your work clearly. Make sure you use **limit notation** where required; an answer that does not use proper notation will not receive full credit. Use = to show things are equal. If you use L'Hôpital's rule, write $\overset{H}{=}$ or $\overset{L'H}{=}$ to indicate where you are applying it.

a. $\displaystyle\lim_{x\to\infty} \frac{\ln(2x^4+3)}{x^4}$

b. $\displaystyle\lim_{x\to 1} \frac{\sqrt{2-x}-x}{x-1}$

8. (12 points)

Compute the following **derivatives**. Show your work. You do NOT need to simplify your answer. Your answer should start $f'(x)$, $\frac{df}{dx}$ etc.

a. $f(\theta) = \ln(\sec\theta + \cot\theta)$

b. $g(x) = \dfrac{\sqrt{3}}{4} + \dfrac{\sqrt{x}}{5} - \dfrac{5}{\sqrt{x}}$

c. $h(x) = \dfrac{e^{3x}}{\sin(x)}$

9. (11 points)

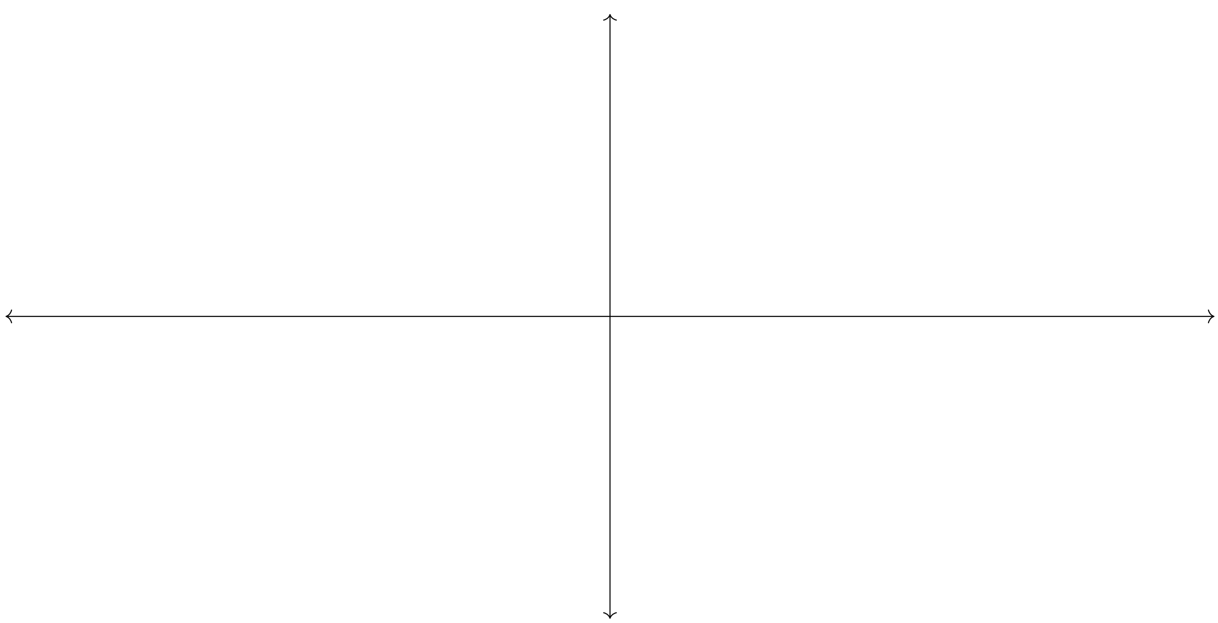
Sketch a graph of a function $h(x)$ that satisfies all of the following properties.

After drawing the graph:

- **Label** on the graph the following things, if they exist, by drawing a point on the graph and labeling: any local maximums by writing LOCAL MAX, local minimums by writing LOCAL MIN, inflection points by writing IP
- **Draw** any horizontal and vertical asymptotes with dashed lines and **label** them with their equation.
- **Mark** any important x-values and y-values (with numbers) on the x- and y-axes.

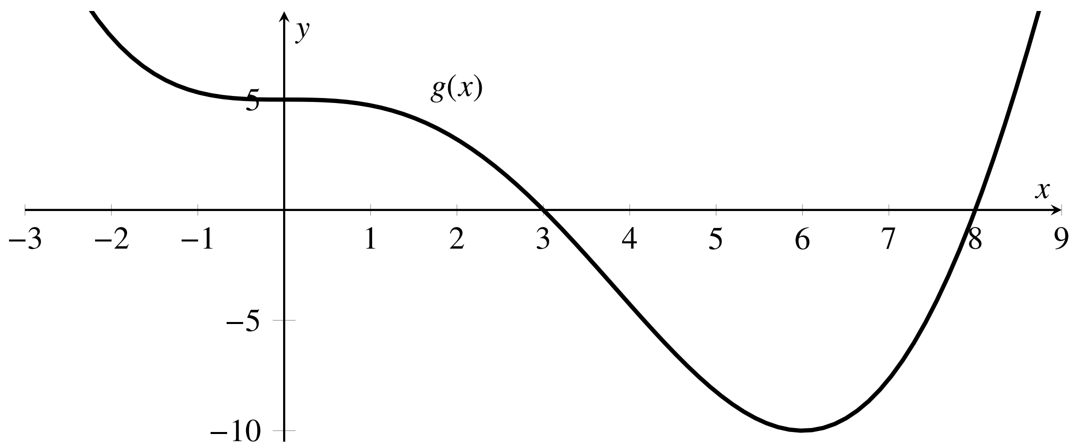
Properties:

- The domain of $h(x)$ is $(-\infty, \infty)$
- $h(0) = 0$ and $h(2) = 2$
- $h'(x) < 0$ on the interval $(-\infty, 0) \cup (2, \infty)$
- $h'(x) > 0$ on the interval $(0, 2)$
- $h''(x) < 0$ on the interval $(1, 3)$
- $h''(x) > 0$ on the interval $(-\infty, 1) \cup (3, \infty)$
- $\lim_{x \to -\infty} h(x) = \infty$
- $\lim_{x \to \infty} h(x) = 0$

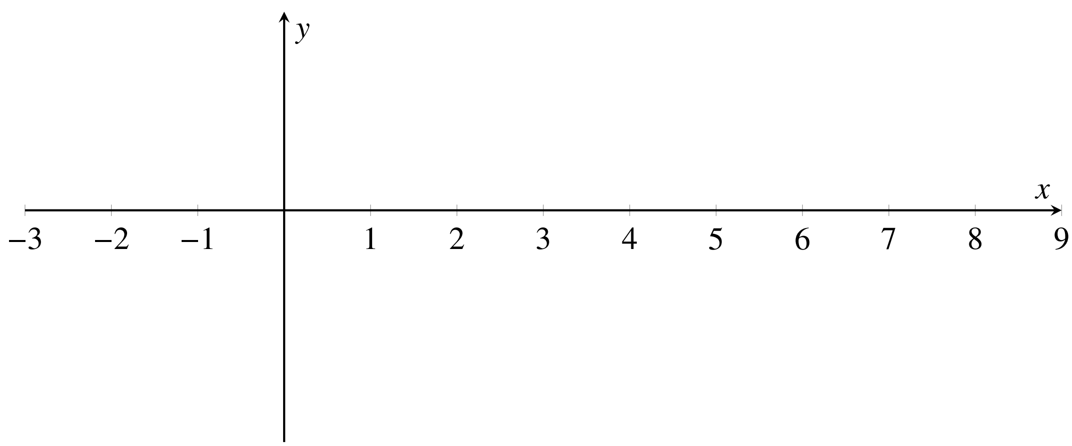


10. (4 points)

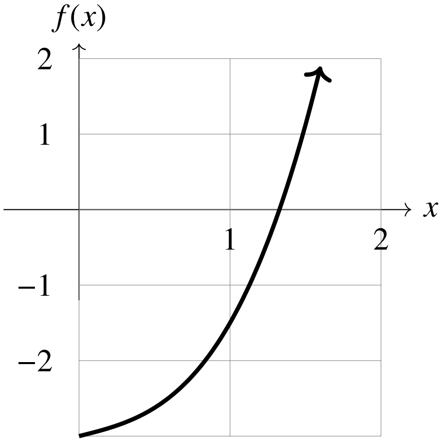
The graph of a function $g(x)$ is shown below.



Sketch the graph of its derivative $g'(x)$ on the axes below.



Extra Credit (5 points) A portion of the graph of the function $f(x) = -3 + x/2 + x^3$ is shown below.



a. Suppose Newton's method is used to find an approximate solution to $f(x) = 0$ from an initial guess of $x_1 = 1$. **Sketch** on the graph how the next approximation x_2 will be found, **labeling** its location on the x-axis.

b. If your starting guess is $x_1 = 1$, **compute** x_2.