Spring 2024 Math F251X

Calculus 1: Midterm 2

Name: Solutions

Section: ☐ 9:15am (James Gossell)
☐ 11:45am (Mohamed Nouh)
☐ async (Leah Berman)

Rules:

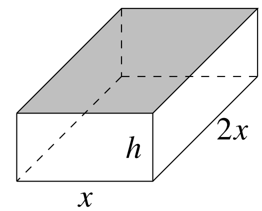
- Partial credit may be awarded, but you must show your work.
- You may have a single handwritten $3'' \times 5''$ notecard, both sides.
- Calculators are **not** allowed.
- Place a box around your $\boxed{\text{FINAL ANSWER}}$ to each question where appropriate.
- Turn off anything that might go beep during the exam.

Good luck!

Problem	Possible	Score
1	12	
2	6	
3	7	
4	10	
5	18	
6	11	
7	12	
8	12	
9	12	
Extra Credit	5	
Total	100	

1. (12 points)

A rectangular storage container is to be constructed to have a fixed volume of 8 m^3. Its bottom and sides are to be made of material that costs \$1 per square meter, while the top is constructed of material that costs \$5/square meter. The length of the base of the box is twice the width.



a. Write h in terms of x using the information in the problem.

$$8 = h \cdot x \cdot 2x \Rightarrow 8 = 2x^2 h \Rightarrow h = \frac{8}{2x^2} = \frac{4}{x^2}$$

b. Write a function for the **total cost of the box** of the box in terms of the variable x.

$$\text{Cost} = \underbrace{2(xh)}_{\text{front/back}} + \underbrace{2(2xh)}_{\text{sides}} + \underbrace{2x^2}_{\text{bottom}} + \underbrace{5(2x^2)}_{\text{top}}$$

$$= 2xh + 4xh + 2x^2 + 10x^2 = 6xh + 12x^2$$

$$\Rightarrow \text{so } C(x) = 6x\left(\frac{4}{x^2}\right) + 12x^2$$

$$= \frac{24}{x} + 12x^2$$

c. Determine the **dimensions** of the box with minimum cost.

Show your work, and use calculus to **justify** that your answer is the minimum. Include units in your final answer. An answer with no clear justification or that does not use calculus techniques will not receive full credit.

$$C(x) = \frac{24}{x} + 12x^2 \Rightarrow$$

$$C'(x) = 24(-x^{-2}) + 24x$$

- Solve $C'(x) = 0 \Rightarrow$

$$-\frac{24}{x^2} + 24x = 0 \Rightarrow$$

$$24x\left(-\frac{1}{x^3} + 1\right) = 0 \Rightarrow$$

$$\boxed{x=0} \text{ or } x^3 = 1 \Rightarrow \boxed{x=1}$$

but $x=0$ is not in the domain ($C(x)$ is undefined at $x=0$).

- So we hope $x=1$ is an absolute min.

Use 2nd deriv test to check:

$$C''(x) = -24(-2x^{-3}) + 24$$

$$= \frac{48}{x^3} + 24$$

$C''(x) > 0$ for $x > 0$ and in particular $C''(1) > 0 \Rightarrow C$ is concave up, so $x=1$ is a min.

- If $x=1$, $h = \frac{8}{2(1)^2} = 4$

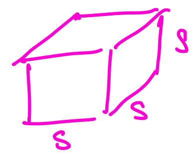
Dimensions: width: 1 length: 2 height: 4

2. (6 points)

Suppose the side length of a cube is measured to be 5 cm, with a possible measurement error of $\pm\frac{1}{10}$ cm.

a. Use linearization or differentials to estimate the possible error in the volume of the cube.

$V = s^3 \Rightarrow \frac{dV}{ds} = 3s^2$



$\frac{\Delta V}{\Delta s} \approx \frac{dV}{ds} \Rightarrow \Delta V \approx \frac{dV}{ds} \Delta s$

When $s = 5$ & $\Delta s = \pm 1/10$,

$\Delta V \approx \pm 3(5)^2 \cdot \frac{1}{10} = \pm \frac{75}{10}$ → So $\Delta V \approx \pm 7.5 \text{ cm}^3$

b. What is the relative error in the volume?

relative error $= \frac{\Delta V}{V} = \frac{75}{10} \cdot \frac{1}{125} = \frac{3 \cdot 5 \cdot 5}{10 \cdot 5 \cdot 5 \cdot 5} = \frac{3}{50} = \frac{6}{100}$

The relative error is $\pm 3/50$, % error is $\pm 6\%$

3. (7 points)

Find the absolute maximum and absolute minimum for the function

$$g(x) = (x-2)(x+3)^2 = x^3 + 4x^2 - 3x - 18$$

on the interval $[-4, 0]$. Justify how you know that you have found the absolute maximum and minimum. If there is no absolute maximum or minimum write "none".

$g'(x) = 3x^2 + 8x - 3$

Solve $g'(x) = 0 \Rightarrow 3x^2 + 8x - 3 = 0$

$\Rightarrow (3x - 1)(x + 3) = 0$

$\Rightarrow x = 1/3$ or $x = -3$ are critical points.

Only $x = -3$ is in $[-4, 0]$

Test critical points and endpoints:

$g(-4) = (-4-2)(-4+3)^2$
$= (-6)(-1)^2 = -6$

$g(-3) = (-3-2)(-3+3)^2$
$= 0$ MAX

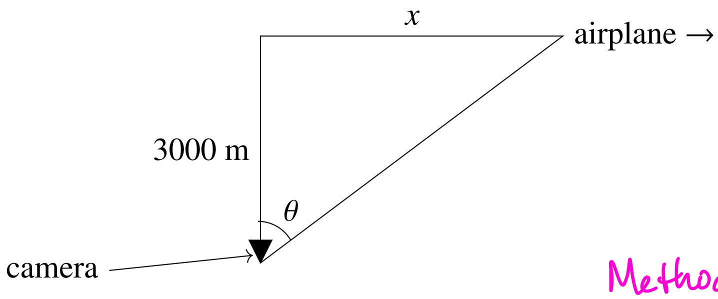
$g(0) = (0-2)(0+3)^2 = -2(9)$
$= -18$ MIN

Absolute maximum: $y =$ 0 Absoute minimum: $y =$ −18

4. (10 points)

A small airplane is flying horizontally at an altitude of 3000 m at a speed of 450 km/hr = 125 m/s.

At time $t = 0$ it passes above a camera on the ground that is pointed directly up. To keep the airplane in view, the angle of the camera changes. How fast is the angle θ (see diagram) changing when the plane has travelled 4000 meters from the spot directly above the camera? Give your answer as an exact number, and include units.



Method #1

$$\tan\theta = \frac{x}{3000} \Rightarrow \theta = \arctan\left(\frac{x}{3000}\right)$$

$$\Rightarrow \frac{d\theta}{dt} = \frac{1}{1+\left(\frac{x}{3000}\right)^2}\left(\frac{1}{3000}\right)\cdot\frac{dx}{dt}$$

Know $\frac{dx}{dt} = 125$ m/s. Want $\frac{d\theta}{dt}$ when $x = 4000$ m

So $\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{4000}{3000}\right)^2}\left(\frac{1}{3000}\right)\cdot 125$

$$= \frac{1}{1+\left(\frac{4}{3}\right)^2}\cdot\frac{1}{3000}\cdot 125 = \frac{1}{\frac{9+16}{9}}\cdot\frac{1}{3000}\cdot 125$$

$$= \frac{9\cdot 125}{25\cdot 3000} = \frac{9\cdot 5}{3000} = \frac{3\cdot 5}{1000} = \boxed{\frac{3}{500}\text{ rad/s}}$$

Method #2:

$x = 3000\tan\theta \Rightarrow$

$\frac{dx}{dt} = 3000\sec^2\theta\,\frac{d\theta}{dt}$

want $\frac{d\theta}{dt}$ when $x = 4000$, $\frac{dx}{dt} = 125$

When $x = 4000$, hyp $= 5000$, so $\cos\theta = \frac{3000}{5000} = 3/5$

$\Rightarrow \sec^2\theta = \frac{25}{9}$, So

$$125 = 3000\cdot\frac{25}{9}\frac{d\theta}{dt} \Rightarrow \frac{d\theta}{dt} = \frac{125\cdot 9}{3000\cdot 25} = \frac{5\cdot 9}{3000} = \frac{5\cdot 3}{1000} = \boxed{\frac{3}{500}\text{ rad/s}}$$

5. (18 points)

Answer the questions below about the function $f(x) = \dfrac{45x^4 - 1584x^2}{440\sqrt[3]{x}}$. It is a fact that after simplification,

$$f'(x) = \frac{3x^3 - 48x}{8\sqrt[3]{x}}, \quad \text{and } f''(x) = \frac{x^2-4}{\sqrt[3]{x}}$$

You must show your work and justify your conclusion with a few words or a computation. Make sure someone else can follow your work.

a. What is the domain of $f(x)$? (use interval notation) $(-\infty, 0) \cup (0, \infty)$

b. Determine the intervals where f is **increasing** and where f is **decreasing**. Show your work.

$3\overline{)48}$, $\frac{3}{18}$

$f'(x) = 0 \Rightarrow \dfrac{3x^3 - 48x}{8\sqrt[3]{x}} = 0$

$f'(x)$ DNE at $x = 0$

$\Rightarrow 3x(x^2 - 16) = 0 \Rightarrow \boxed{x = 0 \text{ or } x = 4 \text{ or } x = -4} \leftarrow$ critical points

x	-5	-4	-1	0	1	4	5
f'	+	0	−	DNE	−	0	+
f	↗	MAX	↘		↘	MIN	↗

$f'(-5) = \dfrac{3(-5)((-5)^2 - 16)}{\sqrt[3]{-5}} = \dfrac{(-)(25-16)}{-} = +$

$f'(-1) = \dfrac{3(-1)((-1)^2 - 16)}{\sqrt[3]{-1}} = \dfrac{(-)(-)}{(-)} = -$

$f'(1) = \dfrac{3(1)(1-16)}{\sqrt[3]{1}} = -$

$f'(5) = \dfrac{3(5)(25-16)}{\sqrt[3]{5}} = +$

Increasing: $(-\infty, -4) \cup (4, \infty)$ Decreasing: $(-4, 0) \cup (0, 4)$ (If none write "none".)

c. Fill in the blanks: $f(x)$ has a local maximum at $x =$ -4 and a local minimum at $x =$ 4. (If none, write "none".)

d. Find all intervals where f is **concave up** and where f is **concave down**. Show your work.

$f''(x) = \dfrac{x^2 - 4}{\sqrt[3]{x}} = \dfrac{(x-2)(x+2)}{\sqrt[3]{x}}$ $f''(x) = 0 \Rightarrow x = 2$ or $x = -2$ $f''(x)$ DNE at $x = 0$.

$f''(-3) = \dfrac{(-5)(-1)}{\sqrt[3]{-3}} = \dfrac{+}{-} = -$ $f''(1) = \dfrac{(-2)(3)}{\sqrt[3]{1}} = \dfrac{-}{+} = -$

x	-3	-2	-1	0	1	2	3
f''	−	0	+	DNE	−	0	+
f	∩		∪		∩		∪

$f''(-1) = \dfrac{(-3)(1)}{\sqrt[3]{-1}} = \dfrac{-}{-} = +$ $f''(3) = \dfrac{(1)(5)}{\sqrt[3]{3}} = +$

Concave up: $(-2, 0) \cup (2, \infty)$ Concave down: $(-\infty, -2) \cup (0, 2)$ (If none write "none".)

e. Fill in the blanks: $f(x)$ has (an) inflection point(s) at $x =$ $-2,\ 2$. (If none, write "none".)

note f is not defined at $x = 0$, which is actually an asymptote of the function.

6. (11 points)

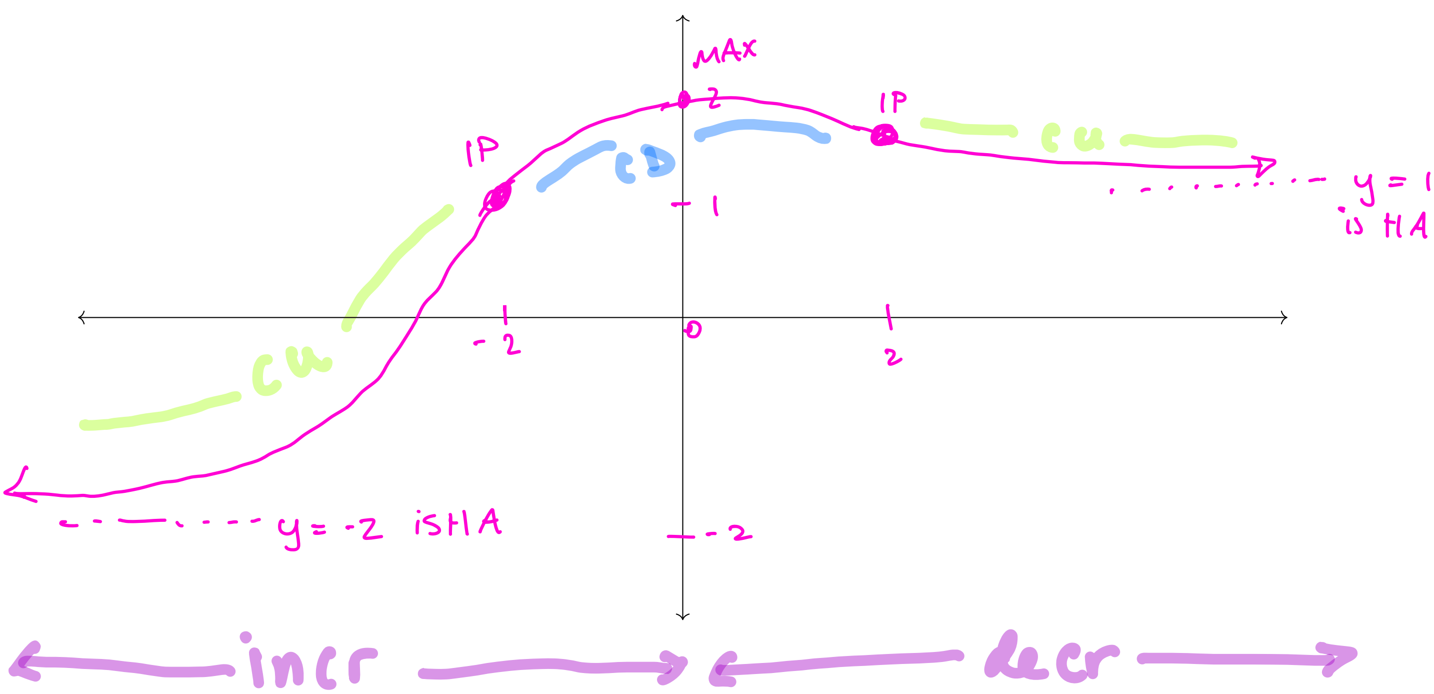
Sketch a graph of a function $h(x)$ that satisfies all of the following properties.

After drawing the graph:

- **Label** on the graph the following things, if they exist, by drawing a point on the graph and labeling: any local maximums by writing LOCAL MAX, local minimums by writing LOCAL MIN, inflection points by writing IP
- Draw any horizontal and vertical asymptotes with dashed lines and **label** them with their equation.
- Mark any important x-values and y-values on the x- and y-axes.

Properties:

- The domain of $h(x)$ is $(-\infty, \infty)$.
- $h(0) = 2$.
- $h'(x) > 0$ when $x < 0$.
- $h'(x) < 0$ when $x > 0$.
- $h''(x) > 0$ when $x < -2$ or $x > 2$. CU CU
- $h''(x) < 0$ when $-2 < x < 2$ CD
- $\lim_{x \to -\infty} h(x) = -2$ y = -2 is HA
- $\lim_{x \to \infty} h(x) = 1$ y = 1 is HA



7. (12 points)

Evaluate the following limits. **Show your work**, uncluding appropriate use of limit notation. If you use L'Hôpital's rule, you must indicate where you are using it by writing $\stackrel{H}{=}$ or $\stackrel{L'H}{=}$ or something similar. Use ∞ or $-\infty$ where appropriate, and if the limit does not exist, write DNE and provide a justification.

a. $\lim_{x\to 4} \frac{\ln(x-3)}{\sqrt{x}-2}$ type $\frac{\ln(4-3)}{\sqrt{4}-2} = \frac{0}{0}$

$\stackrel{L'H}{=} \lim_{x\to 4} \frac{\frac{1}{x-3}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to 4} \frac{2\sqrt{x}}{x-3} = \frac{2(2)}{4-3} = \frac{4}{1} = 4$

b. $\lim_{x\to 0} \frac{x^3-x^2}{3\cos x - 3}$ type $\frac{0-0}{3-3} = \frac{0}{0}$

$\stackrel{L'H}{=} \lim_{x\to 0} \frac{3x^2-2x}{-3\sin x}$ type $\frac{0-0}{-3(0)} = \frac{0}{0}$

$\stackrel{L'H}{=} \lim_{x\to 0} \frac{6x-2}{-3\cos(x)} = \frac{6(0)-2}{-3(1)} = \frac{-2}{3}$

c. $\lim_{x\to\infty} \frac{e^{-x}+2}{\sec\left(\frac{1}{x}\right)}$

$= \lim_{x\to\infty} (e^{-x}+2)\cos\left(\frac{1}{x}\right) = 2\cos(0) = 2$

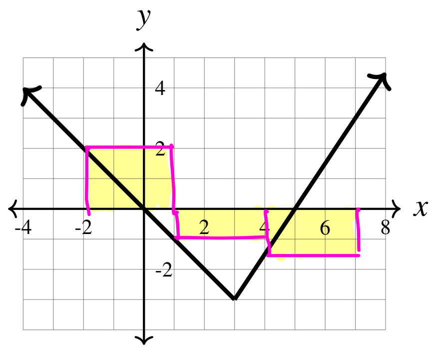
d. $\lim_{x\to\infty} \left(\frac{12x^{10}+8x^2-2}{7x^9+11x^4+6}\right) \frac{1/x^9}{1/x^9} = \lim_{x\to\infty} \frac{12x + \underbrace{8/x^7}_{\to 0} - \underbrace{2/x^9}_{\to 0}}{7 + \underbrace{11/x^5}_{\to 0} + \underbrace{6/x^9}_{\to 0}} = \infty$

8. (12 points)

A portion of the piecewise-linear function

$$f(x) = \begin{cases} -x & x \le 3 \\ \frac{3}{2}(x-5) & x > 3 \end{cases}$$

is graphed below.



a. We want to approximate $\int_{-2}^{7} f(x)\,dx$ using three left-hand rectangles.

i. **Draw** the three **left-hand** rectangles on the graph. Lightly shade them in. Make sure I can tell where your rectangles are and try to be reasonably precise.

ii. Now **approximate** $\int_{-2}^{7} f(x)\,dx$ using the three **left-hand** rectangles. (Your answer should be a number.) Show some work.

$f(-2) = 2$

$f(1) = -1$

$f(4) = \frac{3}{2}(4-5) = -\frac{3}{2}$

width $= 3$

So $\int_{-2}^{7} f(x)dx \approx 3(2) + 3(-1) + 3(-3/2)$

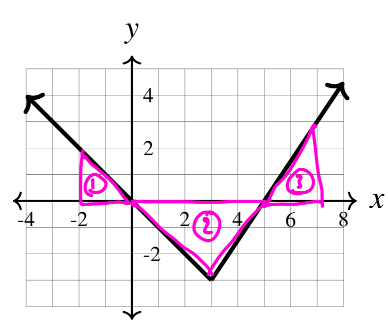
$= 6 - 3 - 9/2 = 3 - 9/2 = 6/2 - \frac{9}{2} = \boxed{-3/2}$

b. Use geometry to compute $\int_{-2}^{7} f(x)\,dx$ exactly. Show your work.

area ① $= \frac{1}{2}(2)(2) = 2$

area ② $= 1/2(5)(-3) = \frac{-15}{2}$

area ③ $= \frac{1}{2}(2)(3) = 3$



total area $= \int_{-2}^{7} f(x)dx = 2 - \frac{15}{2} + 3 = 5 - \frac{15}{2} = \frac{10}{2} - \frac{15}{2}$

$= \frac{-5}{2}$

9. (12 points)

Answer the following, showing work where necessary.

a. $\int 4x - (\sec x)^2 \, dx$ (give the most generic answer)

$= \frac{4x^2}{2} - \tan x + C$

$= 2x^2 - \tan x + C$

Check: $\frac{d}{dx}(\text{stuff}) = 4x - \sec^2(x)$

b. $\int \frac{x^4 - 3x^2 + 5}{x^4} \, dx$ (give the most generic answer)

$= \int 1 - 3x^{-2} + 5x^{-4} \, dx = x - \frac{3x^{-1}}{-1} + \frac{5x^{-3}}{-3} + C$

$= x + \frac{3}{x} - \frac{5}{3x^3} + C$

Check: $\frac{d}{dx}(\text{stuff}) = 1 - 3x^{-2} - \frac{5}{3}(-3x^{-4}) = 1 - 3x^{-2} + 5x^{-4}$

c. The **velocity** $v(t)$ of a particle is given by the function

$$v(t) = \sin(t) - \cos(t)$$

and it has a position function $s(t)$. At time $t = 0$, $s(0) = 1$. Determine the position function $s(t)$.

$s(t) = \int v(t)\,dt = -\cos(t) - \sin(t) + C$

but $s(0) = 1 \Rightarrow -\cos(0) - \sin(0) + C = 1 \Rightarrow -1 - 0 + C = 1 \Rightarrow C = 2$

So $s(t) = -\cos(t) - \sin(t) + 2$.

$\boxed{\text{Extra Credit}}$ (5 points) The growth multiplier over a year for a savings account with an interest rate of 10% compounded x times per year is given by

$$G(x) = \left(1 + \frac{0.1}{x}\right)^x.$$

For example, $G(1) = 1.1$ means that you will multiply your money by 1.1 if interest is compounded once per year. $G(2) = 1.1025$ means that you will multiply your money by 1.1025 if interest is compounded twice per year.

Calculate $\lim_{x\to\infty} G(x)$. What does this value mean in the context of the problem?

$$\ln\left(\lim_{x\to\infty} G(x)\right) = \lim_{x\to\infty} \ln(G(x)) = \lim_{x\to\infty} \ln\left(\left(1 + \frac{0.1}{x}\right)^x\right)$$

$$= \lim_{x\to\infty} x \ln\left(1 + \frac{0.1}{x}\right) \leftarrow \text{type } 0\cdot\infty$$

$$= \lim_{x\to\infty} \frac{\ln\left(1 + \frac{0.1}{x}\right)}{1/x} \quad \text{type } 0/0$$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{\left(\frac{1}{1+\frac{0.1}{x}}\right)\left(\frac{\cancel{-}0.1}{\cancel{x^2}}\right)}{\frac{\cancel{-1}}{\cancel{x^2}}}$$

$$= \lim_{x\to\infty} \frac{0.1}{1 + \frac{0.1}{x}} = 0.1$$

So $\lim_{x\to\infty} G(x) = e^{0.1}$

This means that if your money is compounded continuously, then at the end of the year your money will have grown by a factor of $e^{0.1}$ (≈ 1.10517) $\leftarrow$ which is a little more than if you compounded once or twice a year!